Analytic Solution for the Nucleolus of a Three-Player Cooperative Game[1]

Mingming Leng[2]
Department of Computing and Decision Sciences
Lingnan University
8 Castle Peak Road, Tuen Mun
Hong Kong
(mmleng@ln.edu.hk)

Mahmut Parlar[3]
DeGroote School of Business
McMaster University
Hamilton, Ontario L8S 4M4
Canada
(parlar@mcmaster.ca)

October 2009
Revised March 2010

[1]The authors wish to thank the associate editor and two anonymous referees for their insightful comments that helped improve the paper.
[2]Research supported by the General Research Fund of the Hong Kong Research Grants Council under Research Project No. LU340409.
[3]Research supported by the Natural Sciences and Engineering Research Council of Canada.

Abstract

The nucleolus solution for cooperative games in characteristic function form is usually computed numerically by solving a sequence of linear programming (LP) problems, or by solving a single, but very large-scale, LP problem. This paper proposes an algebraic method to compute the nucleolus solution analytically (i.e., in closed-form) for a three-player cooperative game in characteristic function form. We first consider cooperative games with empty core and derive a formula to compute the nucleolus solution. Next, we examine cooperative games with non-empty core and calculate the nucleolus solution analytically for five possible cases arising from the relationship among the value functions of different coalitions.

Key words: Three-player cooperative game in characteristic function form, nucleolus, linear programming.

1 Introduction

Cooperative game theory studies situations involving multiple players who can cooperate and take joint actions in a coalition to increase their "wealth." The important problem of allocating the newly accrued wealth among the cooperating players in a fair manner has occupied game theorists since the 1940s. More than a dozen alternate solution concepts have been proposed to determine the allocation but only a few of these concepts have received the most attention. Von Neumann and Morgenstern [21] who were the originators of multiperson cooperative games proposed the first solution concept for such games known as the *stable set*. However, due to the theoretical and practical difficulties associated with it, the stable set concept fell out of favour. In 1953, Gillies [6] introduced the concept of *core* as the set of all undominated payoffs (i.e., imputations) to the players satisfying rationality properties. Even though the core has been found useful in studying economic markets, it does not provide a unique solution to the allocation problem. Also in 1953, Shapley [18] wrote three axioms which would capture the idea of a fair allocation of payoffs and developed a simple, analytic, expression to calculate the payoffs. *Shapley value* can be computed easily by using a formula regardless of whether or not the core is empty. However, when the core is non-empty, Shapley value may not be in the core and under some conditions the allocation scheme in terms of Shapley value may result in an unstable grand coalition.

An alternative solution concept known as the *nucleolus* was introduced by Schmeidler [17] in 1969 who proposed an allocation scheme that minimizes the "unhappiness" of the most unhappy player. Schmeidler [17] defines "unhappiness" (or, "excess") of a coalition as the difference between what the members of the coalition could get by themselves and what they are actually getting if they accept the allocations suggested by a solution. It was shown by Schmeidler [17] that if the <R1.3.1
core for a cooperative game is non-empty, then the nucleolus is always located inside the core and thus assures the stability of the grand coalition. Unfortunately, unlike the Shapley value, there exists no closed-form formula for the nucleolus solution which has to be computed numerically in an iterative manner by solving a series of linear programming (LP) problems, or by solving a very large-scale LP problem (see, for example, Owen [14] and Wang [22] for textbook descriptions of these methods). The objective of this paper is to present analytic expressions to calculate the nucleolus solution directly without the need for iterative calculations that involve the solution of linear programs.

Nucleolus solution is an important concept in cooperative game theory even though it is not <R1.3.2
easy to calculate. As Maschler et al. [11, p. 336] pointed out, the nucleolus satisfies some desirable properties—e.g., it always exists uniquely in the core if the core is non-empty, and is therefore <R2.1
considered an important fair division scheme. As a consequence, some researchers have used this concept to analyze business and management problems; but, due to the complexity of calculations, the nucleolus has not been extensively used to solve allocation-related problems. As an early publication for the application of the nucleolus, Barton [1] suggested the nucleolus solution as a tool for accounting decision makers to allocate joint costs among entities who share a common resource. Barton showed that using the nucleolus for this allocation problem can reduce the possibility that

one or more entities may wish to withdraw from the resource-sharing arrangement.

We should point out that in Barton's cost allocation problem, if the cost for running the common <R1.4 resource increases, then the nucleolus solution may suggest a lower cost allocated to some entities which is counter-intuitive. This is possible because, as Megiddo [12] showed, the nucleolus is not always monotonic. For Barton's problem in [1], the monotonicity of a solution means that, if the cost incurred by each possible coalition—in which all entities share the common resource—increases, then the cost allocation to each entity should not. It has been shown that there are other solution concepts that satisfy the monotonicity property and may be used instead of the nucleolus. For example, Young [23] proved that the Shapley value is a unique, monotonic solution, even though, as pointed out above, it may not be in the core if the core is non-empty. In [8], Grotte normalized the nucleolus (by dividing the "excess" of each coalition by the number of players in the coalition) and correspondingly, introduced the new concept "*per capita (normalized)* nucleolus" as an alternative to the original nucleolus solution. Grotte showed that the per capita nucleolus is monotonic and also always exists in the core if the core is non-empty. Thus, for some cost-sharing problems such as that in Barton [1], the per capita nucleolus may be better than the nucleolus solution; but, we note that the calculation for the per capita nucleolus could be even more complicated than that for the nucleolus. For other publications concerning the applications of the nucleolus, see, e.g., Du et al. [4], Gow and Thomas [7], and Leng and Parlar [10].

An n-player game in characteristic-function form (as originally formulated by von Neumann and Morgenstern [21, Ch. VI]) is defined by the set $N = \{1, 2, \ldots, n\}$ and a function $v(\cdot)$ which, for any subset (i.e., coalition) $S \subseteq N$ gives a number $v(S)$ called the value of S (see, also, Straffin [20, Ch. 23]). The characteristic value of the coalition S, denoted by $v(S)$, is the payoff that all players in the coalition S can jointly obtain. For a characteristic function game (N, v), let x_i represent an imputation (i.e., a payoff) for player $i = 1, 2, \ldots, n$. The nucleolus solution is defined as an n-tuple imputation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that the excess ("unhappiness") $e_S(\mathbf{x}) = v(S) - \sum_{i \in S} x_i$ of any possible coalition S cannot be lowered without increasing any other greater excess; see, Schmeidler [17]. With this definition, we find that the nucleolus of a cooperative game is a solution concept that makes the largest unhappiness of the coalitions as small as possible, or, equivalently, minimizes the worst inequity. In the sequential LP method that is based on lexicographic ordering (Maschler et al. [11]), to find the nucleolus solution we first reduce the largest excess $\max\{e_S(x),$ for all $S \subseteq N\}$ as much as possible, then decrease the second largest excess as much as possible, and continue this process until the n-tuple imputation $\mathbf{x}$ is determined.

Since the introduction of the nucleolus solution in 1969, many researchers have proposed alternative approaches to compute the nucleolus. As we indicated above, one of the most popular methods to compute the nucleolus is to solve a series of linear programming (LP) problems. In the past four decades, a number of publications have presented different LP algorithms (some using the sequential LP method, others using a single, but very large, LP formulation) to compute the nucleolus solution. In Table 1, we review these publications in chronological order, and briefly describe their contributions to the LP method for finding the nucleolus.

Year	Author(s)	Brief Description of Major Algorithms in the LP Method
1972	Kohlberg [9]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with n variables and $(2^n)!$ constraints.
1974	Owen [13]	When the set of payoff vectors is a polytope, the nucleolus can be obtained as the solution of a single LP problem with $2^{n+1}+n$ variables and 4^n+1 constraints.
1979	Maschler, Peleg and Shapley [11]	The nucleolus was characterized as the lexicographic center of a cooperative game, and it can be found by solving a series of $O(4^n)$ minimization LP problems with constraint coefficients of either -1, 0 or 1.
1981	Behringer [2]	Simplex based algorithm developed for general lexicographically extended linear maxmin problems to find the nucleolus by solving a sequence of $O(2^n)$ LP problems.
1981	Dragan [3]	Using the concept of coalition array, linear programs with only $O(n)$ rows and $O(2^n)$ columns are used to find the nucleolus solution.
1991	Sankaran [16]	Algorithm to find the nucleolus solution by solving a sequence of $O(2^n)$ LP problems. However, this method needs more constraints than in Behringer [2].
1994	Solymosi and Raghavan [19]	Algorithm to determine the nucleolus of an assignment game. In an (m,n)-person assignment game, the nucleolus is found in at most $m(m+3)/2$ steps, each one requiring at most $O(mn)$ elementary operations.
1996	Potters, Reijnierse and Ansing [15]	The nucleolus solution can be found by solving at most $n-1$ linear programs with at most 2^n-1 rows and 2^n+n-1 columns.
1997	Fromen [5]	By utilizing Behringer's algorithm [2], the number of LP problems to find the nucleolus is reduced to $O(n)$.

Table 1: A brief review of important algorithms to compute the nucleolus using the LP method.

The description of the methods to find the nucleolus as summarized in Table 1 shows that most LP-based methods are iterative in nature and when they are not iterative, the resulting LP can be quite large (as in Kohlberg [9] and Owen [13]). For further discussions regarding these LP methods, see the online Appendix B, in which we compare the LP methods listed in Table 1, and use two examples to illustrate Maschler et al.'s algorithm [11], and Potters et al.'s [15] and Fromen's LP methods [5] that are relatively simpler than the others.

In this paper we focus on three-player cooperative games in characteristic-function form, and present an algebraic method that determines the nucleolus analytically (i.e., using closed-form expressions) without the need for iterative algorithms. Furthermore, we limit our discussion to the <R2.2 case of *superadditive* and *essential* games. [In a superadditive game, $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalitions S and T; and in an essential game, $v(123) > v(1) + v(2) + v(3)$; see, Straffin [20].] This is a reasonable limitation because if a game is not superadditive and/or essential, then the grand coalition will not be stable since the players would be better off by leaving this coalition. Thus, when a game is not superadditive and/or essential, it is unnecessary to examine the problem of fairly allocating the system-wide profit (that is, the characteristic value of grand coalition) among all players. An example of a 3-player game that is not essential is given by Maschler et al. [11] as $[v(\varnothing) \mid v(1), v(2), v(3) \mid v(12), v(13), v(23) \mid v(123)] = [0 \mid 0, 0, 0 \mid 0, 0, 10 \mid 6]$. Here, the grand coalition $\{1,2,3\}$ is not stable since coalition $\{2,3\}$ can gain more if they do not join the grand coalition because $v(23) = 10 > v(123) = 6$.

Without loss of generality, and as justified in Straffin [20, Ch. 23, pp. 152–153], in our three-player superadditive and essential game the characteristic values of the empty and one-player coalitions are assumed zero, i.e., $v(\varnothing) = v(1) = v(2) = v(3) = 0$; the characteristic values of two-player coalitions are non-negative, i.e., $v(ij) \geq 0$, for $i,\ j = 1,\ 2,\ 3,\ i \neq j$; and the characteristic value of the grand coalition $\{123\}$ is positive, i.e., $v(123) > 0$. If this is not the case, then, as discussed in Maschler et al. [11] and demonstrated in Straffin [20, Ch. 23, pp. 153], we can transform any superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. For an example, see the online Appendix A.

The remainder of the paper is organized as follows. In Section 2, we first derive a closed-form algebraic formula to compute the nucleolus solution for three-player characteristic-function cooperative games with empty core. Then, we investigate the computation of the nucleolus when the core of a cooperative game is non-empty, and present five closed-form formulas each arising from the relationship among the value functions of different coalitions. We use two examples to illustrate our algebraic method. In Section 3, we summarize the paper and provide some suggestions for future research.

2 Algebraic Method for Computing the Nucleolus Solution Analytically

In this section, we develop an algebraic method to compute the nucleolus of a three-player cooperative game analytically without the need for linear programming. That is, we derive explicit formulas to compute the nucleolus. We first present our analysis for the relatively simpler case of a cooperative game with empty core. This is followed by the more complicated analysis of the nucleolus computation for cooperative games with non-empty core.

Since we shall minimize the excesses of all possible coalitions to find the nucleolus solution, we first compute these excesses at an imputation $\mathbf{x}$ as follows:

$$e_i(\mathbf{x}) = v(i) - x_i = -x_i, \text{ for } i = 1,2,3, \tag{1}$$

$$e_{ij}(\mathbf{x}) = v(ij) - x_i - x_j = v(ij) - v(123) + x_k, \text{ for } i,j,k = 1,2,3 \text{ and } i \neq j \neq k, \tag{2}$$

$$e_{123}(\mathbf{x}) = v(123) - x_1 - x_2 - x_3 = 0. \tag{3}$$

Note that due to the collective rationality assumption we have $e_{123}(\mathbf{x}) = 0$ in (3); that is, the payoff $v(123)$ of the grand coalition $\{123\}$ is divided to determine three players' payoffs x_1, x_2 and x_3. The collective rationality assumption is then used to find the equalities in (2).

2.1 Algebraic Method for Empty-Core Cooperative Games

We now consider a superadditive and essential cooperative game with empty core, and derive a formula for computing the nucleolus solution.

Theorem 1 If the core of a three-player cooperative game in characteristic function form is empty, then the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ is computed as

$$y_i = \frac{v(123) + v(ij) + v(ik) - 2v(jk)}{3}, \quad \text{for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \tag{4}$$

Proof. See the online Appendix C. ■

We use the formula in Theorem 1 to compute the nucleolus solution for the following cooperative game. In the online Appendix B, we illustrate the solution of the same problem using the more tedious linear programming-based algorithms.

Example 1 Consider the following three-player superadditive and essential cooperative game in characteristic function form: $v(\varnothing) = 0$; $v(i) = 0$, for $i = 1, 2, 3$; $v(12) = 5$, $v(13) = 6$, $v(23) = 8$; $v(123) = 9$. It is easy to show that for this game the core is empty[1]. Using Theorem 1, we compute the nucleolus solution as $y_1 = \frac{1}{3}(5 + 6 + 9 - 2 \cdot 8) = \frac{4}{3}$, $y_2 = \frac{1}{3}(5 + 8 + 9 - 2 \cdot 6) = \frac{10}{3}$ and $y_3 = \frac{1}{3}(6 + 8 + 9 - 2 \cdot 5) = \frac{13}{3}$, which is the same as the result given by solving a series of linear problems in online Appendix B. ◀

2.2 Algebraic Method for Nonempty-Core Cooperative Games

We now derive the formulas that are used to compute the nucleolus solution for a three-player cooperative game with a non-empty core. Since the core is not empty, the nucleolus solution must be in the core (see, for example, Straffin [20, Ch. 23]), and thus, the excesses in (1) and (2) in terms of the nucleolus are non-positive, i.e., $e_j(\mathbf{y}) \leq 0$, for $j = \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$. In order to determine the nucleolus solution, we must first reduce the largest excesses to minimum and then decrease the second largest excess and other excesses. To that end, we first find the necessary and sufficient conditions under which the largest excesses are reduced to the minimum.

Lemma 1 For a three-player cooperative game with a non-empty core, the largest excesses are reduced to minimum if and only if at least one of the following conditions is satisfied:

1. With imputation $\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{1}{3}v(123), \frac{1}{3}v(123), \frac{1}{3}v(123)\right)$, and, $v(123) \geq \max(3v(12), 3v(13), 3v(23))$.
2. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{v(123) + v(12)}{2} - x_2, x_2, \frac{v(123) - v(12)}{2}\right), \tag{5}$$

[1] A simpler method for testing whether the core is empty or not is to solve the following linear program: min x_1 subject to $x_1 + x_2 \geq v(12)$, $x_1 + x_3 \geq v(13)$, $x_2 + x_3 \geq v(23)$, $x_1 + x_2 + x_3 = v(123)$, $x_i \geq 0$, $i = 1, 2, 3$. If the LP has no feasible solution, then the core is empty; otherwise the core is non-empty.

and,

$$\max\left\{v(23), \frac{v(123)-v(12)}{2}\right\} \le x_2 \le \min\left\{v(12), \frac{v(123)+v(12)}{2} - v(13)\right\}. \tag{6}$$

3. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(x_1, \frac{v(123)-v(13)}{2}, \frac{v(123)+v(13)}{2} - x_1\right), \tag{7}$$

and,

$$\max\left\{v(12), \frac{v(123)-v(13)}{2}\right\} \le x_1 \le \min\left\{v(13), \frac{v(123)+v(13)}{2} - v(23)\right\}. \tag{8}$$

4. With imputation

$$\mathbf{x} = (x_1, x_2, x_3) = \left(\frac{v(123)-v(23)}{2}, \frac{v(123)+v(23)}{2} - x_3, x_3\right), \tag{9}$$

and,

$$\max\left\{v(13), \frac{v(123)-v(23)}{2}\right\} \le x_1 \le \min\left\{v(23), \frac{v(123)+v(23)}{2} - v(12)\right\}. \tag{10}$$

5. With imputation

$$x_i = \frac{v(123)+v(ij)+v(ik)-2v(jk)}{3}, \text{ for } i,j,k = 1,2,3 \text{ and } i \neq j \neq k, \tag{11}$$

and,

$$v(123) + v(jk) \le 2[v(ij) + v(ik)], \text{ for } i,j,k = 1,2,3 \text{ and } i \neq j \neq k. \tag{12}$$

Proof. See the online Appendix D. ■

In Lemma 1 we have derived the necessary and sufficient conditions under which the largest excesses are minimized. In order to find the nucleolus solution, we need to reduce the second largest excess and the subsequent excesses to minimum.

Theorem 2 For a three-player, nonempty-core cooperative game in characteristic function form, the nucleolus solution $\mathbf{y} = (y_1, y_2, y_3)$ can be computed as follows:

1. If $v(123) \ge 3v(ij)$, for $i,j = 1,2,3$ and $i \neq j$, then $y_1 = y_2 = y_3 = \frac{1}{3}v(123)$.
2. If $v(123) \ge v(ij) + 2v(ik)$, $v(123) \ge v(ij) + 2v(jk)$ and $v(123) \le 3v(ij)$, for $i,j,k = 1,2,3$ and $i \neq j \neq k$, then $y_i = y_j = \frac{1}{4}[v(123) + v(ij)]$ and $y_k = \frac{1}{2}[v(123) - v(ij)]$.
3. If $v(123) \le v(ij) + 2v(ik)$, $v(123) \ge v(ij) + 2v(jk)$ and $v(ij) \ge v(ik)$, for $i,j,k = 1,2,3$ and $i \neq j \neq k$, then $y_i = \frac{1}{2}[v(ij) + v(ik)]$, $y_j = \frac{1}{2}[v(123) - v(ik)]$, and $y_k = \frac{1}{2}[v(123) - v(ij)]$.
4. If $v(123) + v(ij) \ge 2[v(ik) + v(jk)]$, $v(123) \le v(ij) + 2v(ik)$ and $v(123) \le v(ij) + 2v(jk)$, for

$i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{4}\{v(123) + v(ij) + 2[v(ik) - v(jk)]\},\ y_j = \frac{1}{4}\{v(123) + v(ij) + 2[v(jk) - v(ik)]\},$$
$$y_k = \frac{1}{2}[v(123) - v(ij)].$$

5. If $v(123) + v(ij) \leq 2[v(ik) + v(jk)]$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, then

$$y_i = \frac{1}{3}\{v(123) + v(ij) + v(ik) - 2v(jk)\},\ y_j = \frac{1}{3}\{v(123) + v(ij) + v(jk) - 2v(ik)\},$$
$$y_k = \frac{1}{3}\{v(123) + v(ik) + v(jk) - 2v(ij)\}.$$

Proof. See the online Appendix E. ■

We observe from Theorem 2 that, as the characteristic value of the grand coalition $v(123)$ <R2.6 <AE.2.6 increases, the allocation to one or two players may be decreased. For example, we now consider the second case (in Theorem 2), in which $v(123) \geq v(ij) + 2v(ik)$, $v(123) \geq v(ij) + 2v(jk)$ and $v(123) \leq 3v(ij)$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. For this case, the allocation scheme suggested by the nucleolus solution is given as follow: $y_i = y_j = \frac{1}{4}[v(123) + v(ij)]$ and $y_k = \frac{1}{2}[v(123) - v(ij)]$. Since $v(ij) < v(123)$ for the superadditive and essential game, we find that $y_i = y_j \neq y_k$. If we increase $v(123)$ to a sufficiently large value $v'(123)$ so that the first case in Theorem 2 applies, then we find that the allocation scheme is changed to the following: $y_1 = y_2 = y_3 = \frac{1}{3}v'(123)$. Comparing the new allocation scheme and that obtained before we increase $v(123)$ to $v'(123)$, we find that one or two players may be worse off when the characteristic value of the grand coalition is increased. More specifically, if $v(ij) < \frac{2}{3}v'(123) - v(123)$, then $y_k = \frac{1}{2}[v(123) - v(ij)] > \frac{1}{3}v'(123)$. Because $y_i + y_j + y_k = v(123) < v'(123)$, we find that $y_i = y_j = \frac{1}{4}[v(123) + v(ij)] < \frac{1}{3}v'(123)$. It thus follows that, after the characteristic value of the grand coalition is increased from $v(123)$ to $v'(123)$, player k is worse off and players i and j are better off. We also note that, if $v(ij) > \frac{4}{3}v'(123) - v(123)$, then $y_i = y_j = \frac{1}{4}[v(123) + v(ij)] > \frac{1}{3}v'(123)$ and $y_k = \frac{1}{2}[v(123) - v(ij)] < \frac{1}{3}v'(123)$, which means that player k is better off but players i and j are worse off. This discussion demonstrates that the nucleolus is not always monotonic, as proved by Megiddo [12].

Next, we provide an example to illustrate our analytic results in the above theorem.

Example 2 We now use our algebraic method given in Theorem 2 to solve the following three-player cooperative game: $v(\varnothing) = 0$; $v(i) = 0$, for $i = 1, 2, 3$; $v(12) = 1$, $v(13) = 4$, $v(23) = 3$; $v(123) = 6$. Since the core of this game is non-empty, we use one of the formulas in Theorem 2 to find the nucleolus solution. Since $v(123) = 6 \leq v(13) + 2v(23) = 10$, $v(123) = 6 \geq v(13) + 2v(12) = 6$, $v(13) = 4 \geq v(23) = 3$, the third case (with $i = 3$, $j = 1$ and $k = 2$) in Theorem 2 is eligible to calculate the nucleolus $\mathbf{y} = (y_1, y_2, y_3)$ as $y_1 = [v(123) - v(23)]/2 = 1.5$, $y_2 = [v(123) - v(13)]/2 = 1$ and $y_3 = [v(13) + v(23)]/2 = 3.5$, which is the same as the solution given by the sequential LP method in the online Appendix B. ◀

We have written Maple worksheets which test the emptiness of the core (`CoreTest.mws`), and calculate the nucleolus solution when the core is empty (`Nucleolus-EmptyCore.mws`) and when it is nonempty (`Nucleolus-NonEmptyCore.mws`). These files work with Maple 10, 11 and 12, and they can be downloaded from the authors' web site at `http://www.business.mcmaster.ca/OM/parlar/files/nucleolus/`.

3 Summary and Concluding Remarks

Linear programming plays a prevalent role in computing the nucleolus solution of a cooperative game in the characteristic function form. However, this method requires the solution of a sequence of linear problems, thus making it inconvenient to use. To simplify the computations in calculating the nucleolus, we propose an algebraic method that gives the nucleolus analytically. This paper focuses on a three-player cooperative game. As discussed in Section 2.1, only a single formula is needed for computing the nucleolus solution when the core of a three-player game is empty. In Section 2.2, we derive some formulas each used under three specific conditions. Two examples are presented to illustrate our algebraic method.

References

[1] T. L. Barton. A unique solution for the nucleolus in accounting allocations. *Decision Sciences*, 23(2):365–375, March 1992.

[2] F. A. Behringer. A simplex based algorithm for lexicographically extended linear maxmin problem. *European Journal of Operational Research*, 7:274–283, 1981.

[3] I. Dragan. A procedure for finding the nucleolus of a cooperative n person game. *Mathematical Methods of Operations Research*, 25:119–131, 1981.

[4] S. Du, X. Zhou, L. Mo, and H. Xue. A novel nucleolus-based loss allocation method in bilateral electricity markets. *IEEE Transactions on Power Systems*, 21(1):28–33, February 2006.

[5] B. Fromen. Reducing the number of linear programs needed for solving the nucleolus problem of n-person game theory. *European Journal of Operational Research*, 98:626–636, 1997.

[6] D. B. Gillies. *Some Theorems on n-Person Games*. PhD thesis, Princeton University, Princeton, N.J., 1953.

[7] S. H. Gow and L. C. Thomas. Interchange fees for bank ATM networks. *Naval Research Logistics*, 45(4):407–417, December 1998.

[8] J. H. Grotte. Computation of and observations on the nucleolus and the central games. Master's thesis, Cornell University, 1970. M.Sc. Thesis.

[9] E. Kohlberg. The nucleolus as a solution of a minimization problem. *SIAM Journal on Applied Mathematics*, 23(1):34–39, July 1972.

[10] M. Leng and M. Parlar. Allocation of cost savings in a three-level supply chain with demand information sharing: A cooperative-game approach. *Operations Research*, 57(1):200–213, January–February 2009.

[11] M. Maschler, B. Peleg, and L. Shapley. Geometric properties of the kernel, nucleolus, and related solution concepts. *Mathematics of Operations Research*, 4(4):303–338, November 1979.

[12] N. Megiddo. On the nonmonotonicity of the bargaining set, the kernel and the nucleolus of a game. *SIAM Journal on Applied Mathematics*, 27(2):355–358, September 1974.

[13] G. Owen. A note on the nucleolus. *International Journal of Game Theory*, 3(2):101–103, 1974.

[14] G. Owen. *Game Theory*. Academic Press, New York, 2nd edition, 1982.

[15] J. Potters, J. Reijnierse, and M. Ansing. Computing the nucleolus by solving a prolonged simplex algorithm. *Mathematics of Operations Research*, 21(3):757–768, August 1996.

[16] J. Sankaran. On finding the nucleolus of an n-person cooperative game. *International Journal of Game Theory*, 19:329–338, 1991.

[17] D. Schmeidler. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 17:1163–1170, 1969.

[18] L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton, 1953.

[19] T. Solymosi and T. Raghavan. An algorithm for finding the nucleolus of assignment games. *International Journal of Game Theory*, 23:119–143, 1994.

[20] P. D. Straffin. *Game Theory and Strategy*. The Mathematical Association of America, Washington, D.C., 1993.

[21] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behaviour*. Princeton University Press, Princeton, 1944.

[22] J. Wang. *The Theory of Games*. Oxford University Press, New York, 1988.

[23] H. P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72, June 1985.

Online Appendices

"Analytic Solution for the Nucleolus of a Three-Player Cooperative Game"

Mingming Leng and **Mahmut Parlar**

Appendix A Transformation of a Superadditive and Essential Game to a "Zero-Normalized" Game

We provide an example to show how to transform a superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. Consider the game (N, v) with $[v(\varnothing) \mid v(1), v(2), v(3) \mid v(12), v(13), v(23) \mid v(123)] = [0 \mid 1, 2, 3 \mid 8, 10, 13 \mid 15]$. We can transform (N, v) to the following *strategically equivalent* game (N, v') by subtracting a suitable constant c_i from player i's payoff and (from the value of any coalition containing player i). This gives,

$$\begin{array}{r||r} v'(\varnothing) = 0 & v'(12) = v(12) - v(1) - v(2) = 5 \\ v'(1) = v(1) - 1 = 0 & v'(13) = v(13) - v(1) - v(3) = 6 \\ v'(2) = v(2) - 2 = 0 & v'(23) = v(23) - v(2) - v(3) = 8 \\ v'(3) = v(3) - 3 = 0 & v'(123) = v(123) - v(1) - v(2) - v(3) = 9. \end{array}$$

Using the analytic formula in Section 2.1, the nucleolus solution for this (empty core) game (N, v') is obtained as $\mathbf{y}' = (y_1', y_2', y_3') = (\frac{4}{3}, \frac{10}{3}, \frac{13}{3})$. The nucleolus solution for the original problem is then computed as $\mathbf{y} = (y_1, y_2, y_3) = (\frac{4}{3} + 1, \frac{10}{3} + 2, \frac{13}{3} + 3) = (\frac{7}{3}, \frac{16}{3}, \frac{22}{3})$ which satisfies the collective rationality condition $y_1 + y_2 + y_3 = v(123) = 15$.

Appendix B Sequential LP Method for Computing the Nucleolus Solution

Our brief review presented in Table 1 indicates that, as an early publication on the sequential LP method, Maschler et al. [11] used the concept of lexicographic centre to develop an LP procedure involving $O(4^n)$ LP minimization problems. This LP approach has been adopted by some textbooks (e.g., Wang [22]) as a "typical" method to calculate the nucleolus solution. However, because the LP method in [11] requires solving a large number of linear problems, later researchers have investigated methods to find more efficient LP approach for the calculation of the nucleolus solution.

We see in Table 1 that, immediately after Maschler et al. [11], Behringer [2] reduced the number of LP problems that are needed to find the nucleolus. We also find from Table 1 that, following Behringer [2], others (i.e., Dragan [3], Sankaran [16], and Solymosi and Raghavan [19]) attempted to further improve the LP method; but, they didn't find any method better than Behringer [2]. More specifically, in [3] Dragan's LP approach may need more than $O(2^n)$ linear problems even though this author claimed that only $n-1$ linear programs can be used to find the nucleolus. In addition, the solution found by the LP approach in [3] is actually the prenucleolus rather than the nucleolus solution, as discussed by Potters et al. [15]. Sankaran [16] developed an LP approach which may require the same number of linear problems as in Behringer [2] but needs more constraints. Solymosi and Raghavan's approach in [19] is only applied to a special type of cooperative games (i.e., assignment games).

Potters et al. [15] suggested an LP approach that may reduce the number of linear problems; but, this approach increases the size of each linear problem. From Table 1, we also find that Fromen [5] improved Behringer's algorithm [2] to reduce the number of linear problems without increasing each LP problem's size.

Based on our above discussion, we find that the LP approaches in Potters et al. [15] and Fromen [5] should be two "relatively easy-to-implement" ones compared with other LP methods. Accordingly, in this appendix, we first describe Maschler et al.'s sequential LP approach in Appendix B.1 and present two examples to illustrate this approach; we do this because the approach in [11] is an early one in applying the LP method to the calculation of the nucleolus solution. Then, in Appendices B.2 and B.3, we respectively summarize the LP methods by Potters et al. [15] and Fromen [5], and illustrate these two methods with two numerical examples (that are considered to illustrate Maschler et al.'s LP approach).

Even though this appendix does not contain original theoretical material, we have decided to include it for (i) the clarification of the sequential LP method and (ii) the comparison between the sequential LP method and our algebraic method for a three-player cooperative game. Note that our summary of the LP method is not limited to three-player games but it can be used to find the nucleolus of any cooperative game with $n \geq 3$ players.

B.1 Algorithm for the Sequential LP Method in Maschler et al. [11, 1979]

The existing numerical technique in Maschler et al. [11] requires the sequential solutions of $O(4^n)$ linear programs some of which may exhibit alternative optimal solutions. We next include a careful, step-by-step, description of all the steps involved in Maschler et al.'s algorithm.

In order to find the nucleolus solution using linear programming for a superadditive and essential 0-normalized three-player cooperative game, we develop and solve a sequence of LP problems. To initiate the LP sequences for such a game with imputations $\mathbf{x} = (x_1, x_2, x_3)$, we define $B^0 \equiv \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, as the set of all proper single- and two-player coalitions and $X^0 \equiv \{\mathbf{x} : x_1 + x_2 + x_3 = v(123),\ x_1 \geq v(1),\ x_2 \geq v(2),\ x_3 \geq v(3)\}$, as the set of all feasible allocations/imputations that satisfy, (i) collective rationality, [i.e., $x_1 + x_2 + x_3 = v(123)$], and (ii) individual rationality, [i.e., $x_i \geq v(i)$, $i = 1, 2, 3$]. Note that if the cooperative game had involved n players, then $B^0 \equiv \{S \subset N : S \neq \varnothing \text{ and } S \neq N\}$ and $X^0 \equiv \{\mathbf{x} : \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v(i),\ i = 1, \ldots, n\}$.

Consider the first step in the algorithm and let $k = 1$: First, recall that $e_S(\mathbf{x}) = v(S) - \sum_{i \in S} x_i$ is defined as the "excess/unhappiness" of coalition S with imputation $\mathbf{x}$. We let $\max_{S \in B^0} e_S(\mathbf{x}) = \alpha_1$ as maximal excess and note that $\alpha_1 \geq e_S(\mathbf{x}) = v(S) - x(S)$ for $S \in B^0$ where $x(S) \equiv \sum_{i \in S} x_i$.

Now, formulate the first LP problem as,

$$\min\ z = \alpha_1, \quad \text{subject to} \quad v(S) - x(S) \leq \alpha_1, \quad S \in B^0, \quad \mathbf{x} \in X^0, \quad \alpha_1 \text{ free variable.}$$

Let $\varepsilon^1 = \min_{\mathbf{x} \in X^0} \max_{S \in B^0} e_S(\mathbf{x})$ denote the minimized value of maximal excess and let X^1 denote the allocations found after solving the 1st linear program, i.e., $X^1 \equiv \{\mathbf{x} \in X^0 : \max_{S \in B^0} e_S(\mathbf{x}) = \varepsilon^1\}$. Naturally, there exists no coalition and no allocation in X^1 for which excess exceeds ε^1. However, for some coalitions, excess may be constant and equal to ε^1 for all allocations $\mathbf{x} \in X^1$. Let A^1 denote the set of such coalitions. For the remaining coalitions B^1, there must be some point in X^1 where their excess is less than ε^1. That is,

$$\begin{aligned} A^1 &= \{S \in B^0 : e_S(\mathbf{x}) = \varepsilon^1, \text{ for all } \mathbf{x} \in X^1\}, \\ B^1 &= \{S \in B^0 : e_S(\mathbf{x}) < \varepsilon^1, \text{ for some } \mathbf{x} \in X^1\} = B^0 \setminus A^1. \end{aligned}$$

With this construction, we see that coalitions in A^1 cannot object to imputation X^1 since they all have the same excess and thus they are "neutralized". If B^1 were empty, i.e., if all coalitions belonged to A^1, then X^1 would contain a single point to which there should not be any objections; in this case the single point is *the* nucleolus. For example, if $[v(\varnothing) \mid v(1), v(2), v(3) \mid v(12), v(13), v(23) \mid v(123)] = [0 \mid 0, 0, 0 \mid 2, 2, 2 \mid 6]$, then $x_1 = x_2 = x_3 = 2$, $\varepsilon^1 = -2$ and $A^1 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, thus $B^1 = \varnothing$ and the algorithm would terminate with the nucleolus solution as $y = (2, 2, 2)$.

If B^1 is not empty, we repeat the minimization for step $k = 2$ with respect to those coalitions in B^1 that can still object and solve the 2nd LP problem,

$$\min\ z = \alpha_2, \quad \text{subject to} \quad v(S) - x(S) \leq \alpha_2, \quad S \in B^1, \quad \mathbf{x} \in X^1, \quad \alpha_2 \text{ free variable.}$$

Let ε^2 denote the minimized value of maximal excess and let X^2 denote the allocations found after solving the 2nd linear program, i.e., $X^2 \equiv \{\mathbf{x} \in X^1 : \max_{S \in B^1} e_S(\mathbf{x}) = \varepsilon^2\}$. We also define

$$\begin{aligned} A^2 &= \{S \in B^1 : e_S(\mathbf{x}) = \varepsilon^2, \text{ for all } \mathbf{x} \in X^2\}, \\ B^2 &= \{S \in B^1 : e_S(\mathbf{x}) < \varepsilon^2, \text{ for some } \mathbf{x} \in X^2\} = B^1 \setminus A^2. \end{aligned}$$

We continue in this manner and for $k = 3, \ldots, \kappa$ we let $\varepsilon^k \equiv \min_{\mathbf{x} \in X^{k-1}}\ \alpha_k$, [where $\alpha_k = \max_{S \in B^{k-1}} e_S(\mathbf{x})$] denote the minimized value of maximal excess after solving the kth linear program,

$$\min\ z = \alpha_k, \quad \text{subject to} \quad v(S) - x(S) \leq \alpha_k, \quad S \in B^{k-1}, \quad \mathbf{x} \in X^{k-1}, \quad \alpha_k \text{ free variable.}$$

For $k = 3, \ldots, \kappa$, also define

$$\begin{aligned} A^k &= \{S \in B^{k-1} : e_S(\mathbf{x}) = \varepsilon^k, \text{ for all } \mathbf{x} \in X^k\}, \\ B^k &= \{S \in B^{k-1} : e_S(\mathbf{x}) < \varepsilon^k, \text{ for some } \mathbf{x} \in X^k\} = B^{k-1} \setminus A^k. \end{aligned}$$

$$X^k \equiv \{\mathbf{x} \in X^{k-1} : \max_{S \in B^{k-1}} e_S(\mathbf{x}) = \varepsilon^k\}, \tag{13}$$

Iterations stop when $B^\kappa = \varnothing$ at which stage X^κ will contain a single element, the nucleolus; see Maschler et al. [11] for a proof of the convergence of this algorithm.

To illustrate the sequential LP method as summarized above, recall that the *core* of a three-player game is the set of all imputations $\mathbf{x} = (x_1, x_2, x_3)$ such that $\sum_{i=1}^3 x_i = v(123)$ and for all $S \subseteq N = \{1, 2, 3\}$ we have $\sum_{i \in S} x_i \geq v(S)$. If we cannot find a feasible $\mathbf{x}$ satisfying these conditions, then the core is empty. Next, we use Maschler et al.'s algorithm to re-consider Examples 1 and 2, in order to illustrate this algorithm and compare it with our algebraic method.

B.1.1 Application of Maschler et al.'s Algorithm [11] for the Calculation of Nucleolus Solution of a Cooperative Game with Empty Core

Consider the cooperative game in Example 1. To find the nucleolus solution we start by solving the following LP problem with $B^0 \equiv \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $X^0 \equiv \{\mathbf{x} : x_1 + x_2 + x_3 = v(123),\ x_1 \geq v(1),\ x_2 \geq v(2),\ x_3 \geq v(3)\}$. Note that since α_1 is a free variable, we write $\alpha_1 = \alpha_1' - \alpha_1''$ with $\alpha_1', \alpha_1'' \geq 0$.

$$\begin{aligned} & \min\ z = \alpha_1' - \alpha_1'' \\ \text{subject to} & \\ \{1\}:\ & v(1) - x_1 \leq \alpha_1' - \alpha_1'', \\ \{2\}:\ & v(2) - x_2 \leq \alpha_1' - \alpha_1'', \\ \{3\}:\ & v(3) - x_3 \leq \alpha_1' - \alpha_1'', \\ \{1,2\}:\ & v(12) - (x_1 + x_2) \leq \alpha_1' - \alpha_1'', \\ \{1,3\}:\ & v(13) - (x_1 + x_3) \leq \alpha_1' - \alpha_1'', \\ \{2,3\}:\ & v(23) - (x_2 + x_3) \leq \alpha_1' - \alpha_1'', \\ \mathbf{x} \in X^0:\ & x_1 + x_2 + x_3 = v(123) = 9, \\ \mathbf{x} \in X^0:\ & x_1 \geq v(1) = 0,\ x_2 \geq v(2) = 0,\ x_3 \geq v(3) = 0, \\ \alpha_1:\ & \alpha_1' \geq 0,\ \alpha_1'' \geq 0, \end{aligned} \tag{14}$$

where the $v(\cdot)$ values are given in Example 1.

The *unique* optimal solution for this LP is found as $(\alpha_1')^* = \frac{1}{3}$, $(\alpha_1'')^* = 0$, $\alpha_1^* = \frac{1}{3}$, $x_1^* = 1\frac{1}{3}$, $x_2^* = 3\frac{1}{3}$ and $x_3^* = 4\frac{1}{3}$ which gives $\varepsilon^1 = \alpha_1^* = \frac{1}{3}$. With this optimal solution,the excesses $e_S(\mathbf{x})$ for each coalition are given in the following table:

Coalition:	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$v(S)$	0	0	0	5	6	8
$x(S) = \sum_{i \in S} x_i^*$	$1\frac{1}{3}$	$3\frac{1}{3}$	$4\frac{1}{3}$	$4\frac{2}{3}$	$5\frac{2}{3}$	$7\frac{2}{3}$
$e_S(\mathbf{x}) = v(S) - x(S)$	$-1\frac{1}{3}$	$-3\frac{1}{3}$	$-4\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$,

and,

$$X^1 = \{\mathbf{x} : x_1 = 1\tfrac{1}{3},\ x_2 = 3\tfrac{1}{3},\ x_3 = 4\tfrac{1}{3}\}.$$

Thus, the coalitions with the minimum of the maximum excess $\varepsilon^1 = \frac{1}{3}$ are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$, and as a result $A^1 = \{\{1,2\},\{1,3\},\{2,3\}\}$ so that $B^1 = B^0 \setminus A^1 = \{\{1\},\{2\},\{3\}\}$.

Since $B^1 \neq \varnothing$, we continue with the second step of the algorithm. Because the formulation and computation of the second (and subsequent) LP models are similar to those of the first LP model, we do not provide a detailed discussion again but only summarize the results in Table 2.

Since $B^4 = \varnothing$ as Table 2 indicates, we have found the nucleolus solution as $y = (y_1, y_2, y_3) = (1\frac{1}{3}, 3\frac{1}{3}, 4\frac{1}{3})$.

B.1.2 Application of Maschler et al.'s Algorithm [11] for the Calculation of Nucleolus Solution of a Cooperative Game with a Non-Empty Core

Consider the three-player cooperative game in Example 2. We now solve a sequence of LP problems to compute the nucleolus solution. We model the first LP problem exactly as in (14) but use the new values of $[v(12), v(13), v(23) \mid v(123)]$ as in Example 2. Solving this problem we find $(\alpha_1')^* = 0$, $(\alpha_1'')^* = 1$, $\alpha_1^* = -1$, which gives $\varepsilon^1 = \alpha_1^* = -1$. However, the optimal solution resulting in $\varepsilon^1 = -1$ is *not* unique as we can show using the following arguments: Introducing six surplus variables s_i, $i = 1, \ldots, 6$, one for each inequality in the LP problem, we express the basic variables $(\alpha_1'', x_1, x_2, x_3, s_3, s_4, s_6)$ in terms of the nonbasic variables $(\alpha_1', s_1, s_2, s_5)$ in the final simplex tableau:

$\alpha_1'' = 1 + [\alpha_1'] - \frac{1}{2}[s_2] - \frac{1}{2}[s_5]$	$s_3 = 3 - s_1 + \frac{1}{2}[s_2] + \frac{3}{2}[s_5]$
$x_1 = 1 + s_1 - \frac{1}{2}[s_2] - \frac{1}{2}[s_5]$	$s_4 = s_1 + \frac{1}{2}[s_2] - \frac{1}{2}[s_5]$
$x_2 = 1 + \frac{1}{2}[s_2] - \frac{1}{2}[s_5]$	$s_6 = 1 - s_1 + [s_2] + [s_5]$
$x_3 = 4 - s_1 + [s_5]$	$z = -1 + 0 \cdot [\alpha_1'] + 0 \cdot s_1 + \frac{1}{2}[s_2] + \frac{1}{2}[s_5]$.

It is clear from the expression for z that if either s_2 or s_5 is made basic, then the objective would increase which cannot be allowed, thus we fix $s_2 = s_5 = 0$. Since $\alpha_1'' = 1 + \alpha_1'$, increasing α_1' would also increase α_1'' by the same amount; thus the difference $\alpha_1' - \alpha_1''$ would still be 1, so we also fix $\alpha_1' = 0$, w.l.o.g. Note that the nonbasic variables s_2 and s_5 that must be fixed at 0 are enclosed in a bracket $[\cdot]$ in the above table. However, the nonbasic variable s_1 can be assigned positive values without affecting the value of the objective function provided that the basic variables are not driven to negative levels. Since α_1'' and x_2 do not involve s_1, they are fixed at $\alpha_1'' = 1$ and $x_2 = 1$. Next,

Example	LP Model	Solution	Excesses $e_S(\mathbf{x})$ for $S \in B^1$	A^2, B^2 and X^2
The Second LP Problem				
Example 1 (Empty core)	min $z = \alpha_2' - \alpha_2''$ subject to $\{1\}: \ v(1) - x_1 \leq \alpha_2' - \alpha_2''$, $\{2\}: \ v(2) - x_2 \leq \alpha_2' - \alpha_2''$, $\{3\}: \ v(3) - x_3 \leq \alpha_2' - \alpha_2''$, $\mathbf{x} \in X^1: \ x_1 = 1\frac{1}{3},\ x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}$, $\alpha_2: \ \alpha_2' \geq 0,\ \alpha_2'' \geq 0$.	$(\alpha_2')^* = 0,\ (\alpha_2'')^* = 1\frac{1}{3}$, $\alpha_2^* = -1\frac{1}{3},\ x_1^* = 1\frac{1}{3}$, $x_2^* = 3\frac{1}{3},\ x_3^* = 4\frac{1}{3}$, $\varepsilon^2 = \alpha_2^* = -1\frac{1}{3}$.	$e_{\{1\}}(\mathbf{x}) = -1\frac{1}{3}$, $e_{\{2\}}(\mathbf{x}) = -3\frac{1}{3}$, $e_{\{3\}}(\mathbf{x}) = -4\frac{1}{3}$.	$A^2 = \{\{1\}\}$, $B^2 = B^1 \setminus A^2 = \{\{2\},\{3\}\}$ $X^2 = \{\mathbf{x} : x_1 = 1\frac{1}{3}$, $x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}\}$.
Example 2 (Non-empty core)	min $z = \alpha_2' - \alpha_2''$ subject to $\{1\}: \ v(1) - x_1 \leq \alpha_2' - \alpha_2''$, $\{3\}: \ v(3) - x_3 \leq \alpha_2' - \alpha_2''$, $\{1,2\}: \ v(12) - (x_1 + x_2) \leq \alpha_2' - \alpha_2''$, $\{2,3\}: \ v(23) - (x_2 + x_3) \leq \alpha_2' - \alpha_2''$, $\mathbf{x} \in X^1: \ x_1 \leq 2,\ x_2 = 1,\ x_3 \leq 4,\ x_1 + x_3 = 5$, $\alpha_2: \ \alpha_2' \geq 0,\ \alpha_2'' \geq 0$.	$(\alpha_2')^* = 0,\ (\alpha_2'')^* = \frac{3}{2}$, $\alpha_2^* = -\frac{3}{2},\ x_1 = \frac{3}{2}$, $x_2 = 1,\ x_3 = \frac{7}{2}$, $\varepsilon^2 = \alpha_2^* = -\frac{3}{2}$.	$e_{\{1\}}(\mathbf{x}) = -\frac{3}{2}$, $e_{\{3\}}(\mathbf{x}) = -\frac{7}{2}$, $e_{\{1,2\}}(\mathbf{x}) = -\frac{3}{2}$, $e_{\{2,3\}}(\mathbf{x}) = -\frac{3}{2}$.	$A^2 = \{\{1\},\{1,2\},\{2,3\}\}$, $B^2 = B^1 \setminus A^2 = \{\{3\}\}$, $X^2 = \{\mathbf{x} : x_1 = \frac{3}{2}$, $x_2 = 1,\ x_3 = \frac{7}{2}\}$.
The Third LP Problem				
Example	**LP Model**	**Solution**	**Excesses** $e_S(\mathbf{x})$ **for** $S \in B^2$	A^3, B^3 and X^3
Example 1 (Empty core)	min $z = \alpha_3' - \alpha_3''$ subject to $\{2\}: \ v(2) - x_2 \leq \alpha_3' - \alpha_3''$, $\{3\}: \ v(3) - x_3 \leq \alpha_3' - \alpha_3''$, $\mathbf{x} \in X^2: \ x_1 = 1\frac{1}{3},\ x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}$, $\alpha_3: \ \alpha_3' \geq 0,\ \alpha_3'' \geq 0$.	$(\alpha_3')^* = 0,\ (\alpha_3'')^* = 3\frac{1}{3}$, $\alpha_3^* = -3\frac{1}{3},\ x_1^* = 1\frac{1}{3}$, $x_2^* = 3\frac{1}{3},\ x_3^* = 4\frac{1}{3}$, $\varepsilon^3 = \alpha_3^* = -3\frac{1}{3}$.	$e_{\{2\}}(\mathbf{x}) = -3\frac{1}{3}$, $e_{\{3\}}(\mathbf{x}) = -4\frac{1}{3}$.	$A^3 = \{\{2\}\}$, $B^3 = B^2 \setminus A^3 = \{\{3\}\}$, $X^3 = \{\mathbf{x} : x_1 = 1\frac{1}{3}$, $x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}\}$.
Example 2 (Non-empty core)	min $z = \alpha_3' - \alpha_3''$ subject to $\{3\}: \ v(3) - x_3 \leq \alpha_3' - \alpha_3''$, $\mathbf{x} \in X^2: \ x_1 = \frac{3}{2},\ x_2 = 1,\ x_3 = \frac{7}{2}$ $\alpha_3: \ \alpha_3' \geq 0,\ \alpha_3'' \geq 0$,	$(\alpha_3')^* = 0,\ (\alpha_3'')^* = \frac{7}{2}$, $\alpha_3^* = -\frac{7}{2},\ x_1 = \frac{3}{2}$, $x_2 = 1,\ x_3 = \frac{7}{2}$, $\varepsilon^3 = \alpha_3^* = -\frac{7}{2}$	$e_{\{3\}}(\mathbf{x}) = -\frac{7}{2}$.	$A^3 = \{\{3\}\}$, $B^3 = B^2 \setminus A^3 = \varnothing$, $X^3 = \{\mathbf{x} : x_1 = \frac{3}{2}$, $x_2 = 1,\ x_3 = \frac{7}{2}\}$.
The Fourth LP Problem				
Example	**LP Model**	**Solution**	**Excesses** $e_S(\mathbf{x})$ **for** $S \in B^3$	A^4, B^4 and X^4
Example 1 (Empty core)	min $z = \alpha_4' - \alpha_4''$ subject to $\{3\}: \ v(3) - x_3 \leq \alpha_4' - \alpha_4''$, $\mathbf{x} \in X^3: \ x_1 = 1\frac{1}{3},\ x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}$, $\alpha_4: \ \alpha_4' \geq 0,\ \alpha_4'' \geq 0$.	$(\alpha_4')^* = 0,\ (\alpha_4'')^* = 4\frac{1}{3}$, $\alpha_4^* = -4\frac{1}{3},\ x_1^* = 1\frac{1}{3}$, $x_2^* = 3\frac{1}{3},\ x_3^* = 4\frac{1}{3}$, $\varepsilon^4 = \alpha_4^* = -4\frac{1}{3}$.	$e_{\{3\}}(\mathbf{x}) = -4\frac{1}{3}$.	$A^4 = \{\{3\}\}$, $B^4 = B^3 \setminus A^4 = \varnothing$, $X^4 = \{\mathbf{x} : x_1 = 1\frac{1}{3}$, $x_2 = 3\frac{1}{3},\ x_3 = 4\frac{1}{3}\}$.

Table 2: The LP models (excluding the first LP model) and their solution for Examples 1 and 2. Note that in Example 1, since $B^4 = \varnothing$, we arrive to the nucleolus solution and the algorithm terminates after solving the fourth LP model; and in Example 2, since $B^3 = \varnothing$, we arrive to the nucleolus solution and the algorithm terminates after solving the third LP model.

we solve the five inequalities

$$\begin{array}{l|l} x_1 = 1 + s_1 \geq 0 & s_4 = s_1 \geq 0 \\ x_3 = 4 - s_1 \geq 0 & s_6 = 1 - s_1 \geq 0 \\ s_3 = 3 - s_1 \geq 0 & z = -1, \end{array}$$

simultaneously resulting from the conditions $x_1 \geq 0$, $x_3 \geq 0$, $s_3 \geq 0$, $s_4 \geq 0$ and $s_6 \geq 0$. This gives $0 \leq s_1 \leq 1$ which implies $x_1 \leq 2$, and $x_3 \leq 4$. Thus, noting that $x_2 = 1$ must require $x_1 + x_3 = 5$, we have $X^1 = \{\mathbf{x} : \ x_1 \leq 2, x_2 = 1, \ x_3 \leq 4, \ \ x_1 + x_3 = 5\}$. Also,

$$A^1 = \{\{2\}, \{1,3\}\} \text{ and } B^1 = B^0 \setminus A^1 = \{\{1\}, \{3\}, \{1,2\}, \{2,3\}\},$$

because for coalitions $\{2\}$ and $\{1,3\}$ the excess equals $e_{\{2\}}(x) = 0 - 1 = -1$ and $e_{\{1,3\}}(x) = 4 - 5 = -1$, thus they belong to A^1.

Similar to our discussion in Appendix B.1.1, we present the other LP models and their solutions in Table 2. From Table 2, we find that $B^3 = \varnothing$ and the nucleolus solution is thus $\mathbf{y} = (y_1, y_2, y_3) = (\frac{3}{2}, 1, \frac{7}{2})$.

B.2 Algorithm for the Sequential LP Method in Potters et al. [15]

We first summarize Potters et al.'s sequential LP Method in [15], and use Examples 1 and 2 to illustrate this approach. Let (N, v) denote a superadditive and essential 0-normalized n-player cooperative game with $n \geq 3$. We also denote,

$$\beta \equiv \max_{S \in B^0} v(S) \qquad \text{and} \qquad f_S \equiv \beta - e_S(\mathbf{x}) = \beta - [v(S) - x(S)],$$

where $B^0 = \{S \subset N : S \neq \varnothing \text{ and } S \neq N\}$, as defined in Appendix B.1.

Using the above notations, we summarize Potters et al.'s algorithm as follows:

1. **Initialization**: Let $j = 0$. Arbitrarily take $\gamma \in N$, and consider the following $2^n - 1$ equations:

$$\begin{cases} x(N) = v(N), \\ f_S + x(N \backslash S) = v(N) + \beta - v(S), & \text{if } S \in B^0 \text{ and } \gamma \in S, \\ f_S - x(S) = \beta - v(S), & \text{if } S \in B^0 \text{ but } \gamma \notin S. \end{cases}$$

The above equations can be written as,

$$\Pi_1 \mathbf{x} + \Pi_2 \mathbf{f} = \mathbf{d}, \tag{15}$$

where $\mathbf{x}$ denotes the $n-$dimensional column vector (x_i, for $i \in N$), as defined in Section 1, i.e., $\mathbf{x}$ =(x_i, for $i \in N$). Moreover, in (15), $\mathbf{f}$ denotes the $(2^n - 2)-$dimensional column vector (f_S, for $S \in B^0$), i.e., $\mathbf{f} \equiv$ (f_S, for $S \in B^0$); Π_1 and Π_2 denote the $(2^n - 1) \times n$ coefficient matrix for $\mathbf{x}$ and the $(2^n - 1) \times (2^n - 2)$ coefficient matrix for $\mathbf{f}$, respectively; and $\mathbf{d}$ denotes the $(2^n-1)-$dimensional column vector with the ith element d_i as the constant number at the RHS of the ith equation $(i=1,2,\ldots,2^n-1)$.

2. **Search for the nucleolus solution.** If $B^j=\varnothing$, then the algorithm is terminated, and we can solve the corresponding equations to find the nucleolus; otherwise, if $B^j\neq\varnothing$, then we continue with the following three steps:
   (a) Let u_{1i} and u_{2i} denote the sum of all coefficients of the f_S variables in the ith row of the matrices Π_1 and Π_2, respectively. Solve the following maximization problem,

$$\begin{aligned}\max\quad & t\\ \text{s.t.}\quad & \Pi_1\mathbf{x}+\Pi_2\mathbf{f}+\mathbf{h}t=\mathbf{d};\ t\geq 0,\ \mathbf{x}\geq 0,\ \mathbf{f}\geq 0.\end{aligned}$$

   In the above maximization problem, $\mathbf{h}$ denotes the $(2^n-1)-$dimensional column vector with the ith element $h_i\equiv u_{1i}+u_{2i}$. As Potters et al. [15] suggested, use the simplex method to solve the maximization problem in the tableau, starting with a pivot operation in the $t-$column.
   Then, do the pivot operations *until* there is a single row in which *the coefficient of* t *is* 1 and *the coefficients for other variables (i.e.,* x_i*,* $\forall i\in N$ *and* f_S*,* $\forall S\in B^0$*) are nonnegative*. Assume, w.l.o.g., that the row after the pivot operation is the rth row.
   (b) Do the following operations:
      i. If, for $S\in B^j$, the rth coordinate of the f_s-column (i.e., the column for f_S) is positive, then denote $B^{j+1}=B^j\backslash\{S\}$ and remove the f_s-column.
      ii. If, for $i\in N$, the rth coordination of the x_i-column (i.e., the column for x_i) is positive, then replace the x_i-column by the $0-$column and add the equation: $x_i=0$.
      iii. If $t=d_r$, i.e., the rth row is an elementary row with the coefficient of t as 1, then delete the rth row.
      iv. If, in an elementary row, only the value for the coefficient of f_s is 1, then remove both the elementary row and the f_s-column.
   (c) The resulting tableau is $\Pi_1\mathbf{x}+\Pi_2\mathbf{f}=\mathbf{d}$. Set $j:=j+1$, and go back to examine if $B^j=\varnothing$. If $B^j\neq\varnothing$, then repeat the above three steps.

In [15] Potters et al. proved that, using the above algorithm, we can find the nucleolus solution. Next, we use the algorithm to re-solve the cooperative games in Examples 1 and 2.

B.2.1 Application of Potters et al.'s Algorithm [15] for the Calculation of Nucleolus Solution of a Cooperative Game with Empty Core

Consider the cooperative game in Example 1. We follow Potters et al.'s Algorithm to search for the nucleolus solution as follows:

1. **Initialization**. For this game, $B^0\equiv\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ and $\beta=8$. We

arbitrarily take $\gamma = 1$. We write 7 equations as follows:

$$\begin{cases} x_1 + x_2 + x_3 = 9, \\ f_{12} + x_3 = 12, \\ f_{13} + x_2 = 11, \\ f_1 + x_2 + x_3 = 17, \\ f_{23} - x_2 - x_3 = 0, \\ f_2 - x_2 = 8, \\ f_3 - x_3 = 8, \end{cases}$$

which can be written as $\Pi_1 \mathbf{x} + \Pi_2 \mathbf{f} = \mathbf{d}$, where

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, x_3],\ \mathbf{f} = [f_{12}, f_{13}, f_{23}, f_1, f_2, f_3]; \\ \Pi_1 &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ \mathbf{d} = \begin{bmatrix} 9 \\ 12 \\ 11 \\ 17 \\ 0 \\ 8 \\ 8 \end{bmatrix}. \end{aligned}$$

2. **Search for the nucleolus solution**. Since $B^0 \neq \varnothing$, we consider the following three steps:
   (a) We involve the $t-$column, and have the tableau as,

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	0	0	0	0	0	1	1	1	9	$S = v(123)$
1	1	0	0	0	0	0	0	0	1	12	$S = v(12)$
1	0	1	0	0	0	0	0	1	0	11	$S = v(13)$
1	0	0	0	1	0	0	0	1	1	17	$S = v(1)$
1	0	0	1	0	0	0	0	−1	−1	0	$S = v(23)$
1	0	0	0	0	1	0	0	−1	0	8	$S = v(2)$
1	0	0	0	0	0	1	0	0	−1	8	$S = v(3)$

The first pivot is in the $t-$column. We pivot with the fifth row and thus have,

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	0	0	0	0	0	1	1	1	9	$S=\{1,2,3\}$
0	1	0	-1	0	0	0	0	1	2	12	$S=\{1,2\}$
0	0	1	-1	0	0	0	0	2	1	11	$S=\{1,3\}$
0	0	0	-1	1	0	0	0	2	2	17	$S=\{1\}$
1	0	0	1	0	0	0	0	-1	-1	0	$S=\{2,3\}$
0	0	0	-1	0	1	0	0	0	1	8	$S=\{2\}$
0	0	0	-1	0	0	1	0	0	0	8	$S=\{3\}$

For the second pivot we choose a column with negative fifth coordinate, e.g., the column corresponding to x_2. We pivot with the third row:

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	1	0	$\frac{1}{2}$	$\frac{7}{2}$	$S=\{1,2,3\}$
0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	$\frac{3}{2}$	$\frac{13}{2}$	$S=\{1,2\}$
0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	1	$\frac{1}{2}$	$\frac{11}{2}$	$S=\{1,3\}$
0	0	-1	0	1	0	0	0	0	1	6	$S=\{1\}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{2}$	$\frac{11}{2}$	$S=\{2,3\}$
0	0	0	-1	0	1	0	0	0	1	8	$S=\{2\}$
0	0	0	-1	0	0	1	0	0	0	8	$S=\{3\}$

Next, the third pivot must be in the column corresponding to x_3, because, in the fifth row, the coefficient of x_3 is negative. We pivot with the second row:

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	0	1	0	0	$\frac{4}{3}$	$S=\{1,2,3\}$
0	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	0	1	$\frac{13}{3}$	$S=\{1,2\}$
0	-1	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0	1	0	$\frac{10}{3}$	$S=\{1,3\}$
0	-2	$-\frac{2}{3}$	$\frac{1}{3}$	1	0	0	0	0	0	$\frac{5}{3}$	$S=\{1\}$
1	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	0	0	$\frac{23}{3}$	$S=\{2,3\}$
0	-2	$\frac{1}{3}$	$-\frac{2}{3}$	0	1	0	0	0	0	$\frac{11}{3}$	$S=\{2\}$
0	0	0	-1	0	0	1	0	0	0	8	$S=\{3\}$

Since there is no negative coefficient in the fifth row, we finish Step (a). Let $r=5$.

(b) Because the 5th coordinates of the $f_{12}-$, $f_{13}-$ and $f_{23}-$columns are positive, we denote $B^1 = B^0\backslash\{\{1,2\},\{1,3\},\{2,3\}\} = \{\{1\},\{2\},\{3\}\}$ and remove the $t-$, $f_{12}-$, $f_{13}-$ and $f_{23}-$columns and the fifth row.

(c) The resulting tableau is,

f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	0	1	0	0	$\frac{4}{3}$	$S = v(123)$
0	0	0	0	0	1	$\frac{13}{3}$	$S = v(12)$
0	0	0	0	1	0	$\frac{10}{3}$	$S = v(13)$
1	0	0	0	0	0	$\frac{5}{3}$	$S = v(1)$
0	1	0	0	0	0	$\frac{11}{3}$	$S = v(2)$
0	0	1	0	0	0	8	$S = v(3)$

Since all rows are elementary, the algorithm terminates. We solve the tableau for $\mathbf{x}$, and thus find the nucleolus as $y = (y_1, y_2, y_3) = (\frac{4}{3}, \frac{10}{3}, \frac{13}{3}) = (1\frac{1}{3}, 3\frac{1}{3}, 4\frac{1}{3})$, which is the same as that found in Example 1.

B.2.2 Application of Potters et al.'s Algorithm [15] for the Calculation of Nucleolus Solution of a Cooperative Game with a Non-Empty Core

Consider the cooperative example in Example 2. We follow Potters et al.'s Algorithm to search for the nucleolus solution as follows:

1. **Initialization**. For this game, $B^0 \equiv \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ and $\beta = 4$. We arbitrarily take $\gamma = 1$. We write 7 equations as follows:

$$\begin{cases} x_1 + x_2 + x_3 = 6, \\ f_{12} + x_3 = 9, \\ f_{13} + x_2 = 6, \\ f_1 + x_2 + x_3 = 10, \\ f_{23} - x_2 - x_3 = 1, \\ f_2 - x_2 = 4, \\ f_3 - x_3 = 4, \end{cases}$$

which can be written as $\Pi_1 \mathbf{x} + \Pi_2 \mathbf{f} = \mathbf{d}$, where

$$\mathbf{x} = [x_1, x_2, x_3],\ \mathbf{f} = [f_{12}, f_{13}, f_{23}, f_1, f_2, f_3];$$

$$\Pi_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 6 \\ 9 \\ 6 \\ 10 \\ 1 \\ 4 \\ 4 \end{bmatrix}.$$

2. **Search for the nucleolus solution**. Since $B^0 \neq \varnothing$, we consider the following three steps:

(a) We involve the $t-$column, and have the tableau as,

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	0	0	0	0	0	1	1	1	6	$S = \{1,2,3\}$
1	1	0	0	0	0	0	0	0	1	9	$S = \{1,2\}$
1	0	1	0	0	0	0	0	1	0	6	$S = \{1,3\}$
1	0	0	0	1	0	0	0	1	1	10	$S = \{1\}$
1	0	0	1	0	0	0	0	-1	-1	1	$S = \{2,3\}$
1	0	0	0	0	1	0	0	-1	0	4	$S = \{2\}$
1	0	0	0	0	0	1	0	0	-1	4	$S = \{3\}$

The first pivot is in the $t-$column. We pivot with the fifth row and thus have,

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	0	0	0	0	0	1	1	1	6	$S = \{1,2,3\}$
0	1	0	-1	0	0	0	0	1	2	8	$S = \{1,2\}$
0	0	1	-1	0	0	0	0	2	1	5	$S = \{1,3\}$
0	0	0	-1	1	0	0	0	2	2	9	$S = \{1\}$
1	0	0	1	0	0	0	0	-1	-1	1	$S = \{2,3\}$
0	0	0	-1	0	1	0	0	0	1	3	$S = \{2\}$
0	0	0	-1	0	0	1	0	1	0	3	$S = \{3\}$

For the second pivot we choose a column with negative fifth coordinate, e.g., the column corresponding to x_2. We pivot with the third row:

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	1	0	$\frac{1}{2}$	$\frac{7}{2}$	$S = \{1,2,3\}$
0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	$\frac{3}{2}$	$\frac{11}{2}$	$S = \{1,2\}$
0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	1	$\frac{1}{2}$	$\frac{5}{2}$	$S = \{1,3\}$
0	0	-1	0	1	0	0	0	0	1	4	$S = \{1\}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{2}$	$\frac{7}{2}$	$S = \{2,3\}$
0	0	0	-1	0	1	0	0	0	1	3	$S = \{2\}$
0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$S = \{3\}$

Next, the third pivot must be in the column corresponding to x_3, because, in the fifth row, the coefficient of x_3 is negative. We pivot with the sixth row:

t	f_{12}	f_{13}	f_{23}	f_1	f_2	f_3	x_1	x_2	x_3	d	
0	0	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	1	0	0	2	$S=\{1,2,3\}$
0	1	$-\frac{1}{2}$	1	0	$-\frac{3}{2}$	0	0	0	0	1	$S=\{1,2\}$
0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	1	0	1	$S=\{1,3\}$
0	0	-1	1	1	-1	0	0	0	0	1	$S=\{1\}$
1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	5	$S=\{2,3\}$
0	0	0	-1	0	1	0	0	0	1	3	$S=\{2\}$
0	0	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	1	0	0	0	2	$S=\{3\}$

Since there is no negative coefficient in the fifth row, we finish Step (a). Let $r = 5$.

(b) Because the 5th coordinates of the $f_{13}-$ and f_2-columns are positive, we denote $B^1 = B^0 \backslash \{\{1,3\},\{2\}\} = \{\{1\},\{3\},\{1,2\},\{2,3\}\}$ and remove the $t-$, $f_{13}-$ and f_2-columns and the fifth row.

(c) The resulting tableau is,

f_{12}	f_{23}	f_1	f_3	x_1	x_2	x_3	d	
0	1	0	0	1	0	0	2	$S=\{1,2,3\}$
1	1	0	0	0	0	0	1	$S=\{1,2\}$
0	0	0	0	0	1	0	1	$S=\{1,3\}$
0	1	1	0	0	0	0	1	$S=\{1\}$
0	-1	0	0	0	0	1	3	$S=\{2\}$
0	-1	0	1	0	0	0	2	$S=\{3\}$

3. Since $B^1 \neq \varnothing$, we get the new $t-$column by making the sum of the coefficients of the f_S-columns in each row, and find that,

t	f_{12}	f_{23}	f_1	f_3	x_1	x_2	x_3	d	
1	0	1	0	0	1	0	0	2	$S=\{1,2,3\}$
2	1	1	0	0	0	0	0	1	$S=\{1,2\}$
0	0	0	0	0	0	1	0	1	$S=\{1,3\}$
2	0	1	1	0	0	0	0	1	$S=\{1\}$
-1	0	-1	0	0	0	0	1	3	$S=\{2\}$
0	0	-1	0	1	0	0	0	2	$S=\{3\}$

(a) We pivot with the second row, and obtain,

t	f_{12}	f_{23}	f_1	f_3	x_1	x_2	x_3	d	
0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0	0	$\frac{3}{2}$	$S = \{1,2,3\}$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	$S = \{1,2\}$
0	0	0	0	0	0	1	0	1	$S = \{1,3\}$
0	-1	0	1	0	0	0	0	0	$S = \{1\}$
0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	1	$\frac{7}{2}$	$S = \{2\}$
0	0	-1	0	1	0	0	0	2	$S = \{3\}$

Since there is no negative coefficient in the second row, we finish Step (a). Let $r = 2$.

(b) Because the 2th coordinates of the $f_{12}-$ and $f_{23}-$columns are positive, we denote $B^2 = B^1 \backslash \{\{1,2\},\{2,3\}\} = \{\{1\},\{3\}\}$ and remove the $t-$, $f_{12}-$ and $f_{23}-$columns and the second row.

(c) The resulting tableau is,

f_1	f_3	x_1	x_2	x_3	d	
0	0	1	0	0	$\frac{3}{2}$	$S = \{1,2,3\}$
0	0	0	1	0	1	$S = \{1,3\}$
1	0	0	0	0	0	$S = \{1\}$
0	0	0	0	1	$\frac{7}{2}$	$S = \{2\}$
0	1	0	0	0	2	$S = \{3\}$

Since all rows are elementary, the algorithm terminates. We solve the tableau for $\mathbf{x}$, and thus find the nucleolus as $y = (y_1, y_2, y_3) = (\frac{3}{2}, 1, \frac{7}{2})$, which is the same as that found in Example 2.

B.3 Algorithm for the Sequential LP Method in Fromen [5]

We now summarize Fromen's sequential LP Method in [5], which is then illustrated by using Examples 1 and 2. In [5] Fromen suggested that, if the concept of "matrix rank" is introduced to Behringer's LP approach in [2], then the number of linear problems solved for the nucleolus can be reduced to $O(n)$.

Using our notations in Appendix B.1, we summarize Fromen's algorithm as follows:

1. The first linear problem is developed as,

$$\min\ z = \alpha_1, \quad \text{subject to} \quad v(S) - x(S) \le \alpha_1, \quad S \in B^0, \quad \mathbf{x} \in X^0, \quad \alpha_1 \text{ free variable},$$

where, as defined in Appendix B.1, $B^0 = \{S \subset N : S \neq \varnothing \text{ and } S \neq N\}$ and $X^0 = \{\mathbf{x} : \sum_{i=1}^{n} x_i = v(N),\ x_i \ge v(i),\ i = 1, \ldots, n\}$. We also recall from Appendix B.1 that

$\varepsilon^1 = \min_{\mathbf{x}\in X^0} \max_{S\in B^0} e_S(\mathbf{x})$ denote the minimized value of maximal excess; and,

$$\begin{aligned} X^1 &= \{\mathbf{x} \in X^0 : \max_{S\in B^0} e_S(\mathbf{x}) = \varepsilon^1\}, \\ A^1 &= \{S \in B^0 : e_S(\mathbf{x}) = \varepsilon^1, \text{ for all } \mathbf{x} \in X^1\}, \\ B^1 &= \{S \in B^0 : e_S(\mathbf{x}) < \varepsilon^1, \text{ for some } \mathbf{x} \in X^1\} = B^0 \setminus A^1. \end{aligned}$$

According to Fromen's algorithm (or, Behringer's algorithm), we should determine whether or not the search process terminates and we arrive to the nucleolus solution, before solving the second linear problem. Let Λ^1 denote the matrix of the coefficients of x_i $(i \in N)$ in the following equations: $e_S(\mathbf{x}) = \varepsilon^1$, for all $S \in A^1$. If the rank of Λ^1 is equal to n, then the algorithm terminates with the nucleolus solution as the optimal solution for the first linear problem; otherwise, continue with the second linear problem.

2. Let

$$Z^1 \equiv \{\mathbf{x} : \sum_{i=1}^{n} x_i = v(N);\ x_i \geq v(i),\ i \in B^1;\ e_S(\mathbf{x}) = \varepsilon^1, \text{ for all } S \in A^1\}.$$

Solve the second LP problem,

$$\min\ z = \alpha_2, \quad \text{subject to} \quad v(S) - x(S) \leq \alpha_2, \quad S \in B^1, \quad \mathbf{x} \in Z^1, \quad \alpha_2 \text{ free variable.}$$

Using the optimal solution of the second linear problem, we denote by ε^2 the minimized value of maximal excess, i.e., $\varepsilon^2 = \min_{\mathbf{x}\in Z^1} \max_{S\in B^1} e_S(\mathbf{x})$; and,

$$\begin{aligned} X^2 &= \{\mathbf{x} \in Z^1 : \max_{S\in B^1} e_S(\mathbf{x}) = \varepsilon^2\}, \\ A^2 &= \{S \in B^1 : e_S(\mathbf{x}) = \varepsilon^2, \text{ for all } \mathbf{x} \in X^2\}, \\ B^2 &= \{S \in B^1 : e_S(\mathbf{x}) < \varepsilon^2, \text{ for some } \mathbf{x} \in X^2\} = B^1 \setminus A^2. \end{aligned}$$

Let Λ^2 denote the matrix of the coefficients of x_i $(i \in N)$ in the following equations: $e_S(\mathbf{x}) = \varepsilon^r$, for all $S \in A^r$, $r = 1, 2$. If the rank of Λ^2 is equal to n, then the algorithm terminates with the nucleolus solution as the optimal solution for the second linear problem; otherwise, continue with the third linear problem.

3. We continue in this manner and for $k = 3, \ldots, \kappa$ we let

$$Z^{k-1} \equiv \{\mathbf{x} : \sum_{i=1}^{n} x_i = v(N);\ x_i \geq v(i),\ i \in B^2;\ e_S(\mathbf{x}) = \varepsilon^r, \forall S \in A^r, r = 1, 2, \ldots, k-1\},$$

and, $\varepsilon^k \equiv \min_{\mathbf{x}\in Z^{k-1}} \max_{S\in B^{k-1}} e_S(\mathbf{x})$ denotes the minimized value of maximal excess after solving the kth linear program,

$$\min\ z = \alpha_k, \quad \text{subject to} \quad v(S) - x(S) \leq \alpha_k, \quad S \in B^{k-1}, \quad \mathbf{x} \in Z^{k-1}, \quad \alpha_k \text{ free variable.}$$

For $k = 3, \ldots, \kappa$, also define

$$\begin{aligned} X^k &= \{\mathbf{x} \in Z^{k-1} : \max_{S \in B^{k-1}} e_S(\mathbf{x}) = \varepsilon^k\}, \\ A^k &= \{S \in B^{k-1} : e_S(\mathbf{x}) = \varepsilon^k, \text{ for all } \mathbf{x} \in X^k\}, \\ B^k &= \{S \in B^{k-1} : e_S(\mathbf{x}) < \varepsilon^k, \text{ for some } \mathbf{x} \in X^k\} = B^{k-1} \setminus A^k. \end{aligned}$$

Let Λ^k denote the matrix of the coefficients of x_i $(i \in N)$ in the following equations: $e_S(\mathbf{x}) = \varepsilon^r$, $\forall S \in A^r$, for $r = 1, 2, \ldots, k$. Iterations stop when the rank of Λ^2 is equal to n. Fromen [5] proved the convergence of this algorithm.

Next, we again use Examples 1 and 2 to illustrate the above LP method.

B.3.1 Application of Fromen's Algorithm [5] for the Calculation of Nucleolus Solution of a Cooperative Game with Empty Core

Using the cooperative game in Example 1, we illustrate Fromen's LP method for finding the nucleolus solution of a cooperative game with empty core.

1. We notice that the first linear problem in Fromen's Algorithm is the same as that in Maschler et al.'s algorithm. From Appendix B.1.1, we find that $\varepsilon^1 = \frac{1}{3}$, and

$$\begin{aligned} X^1 &= \{\mathbf{x} : x_1 = 1\tfrac{1}{3},\ x_2 = 3\tfrac{1}{3},\ x_3 = 4\tfrac{1}{3}\}, \\ A^1 &= \{\{1,2\}, \{1,3\}, \{2,3\}\}, \\ B^1 &= \{\{1\}, \{2\}, \{3\}\}. \end{aligned}$$

We can thus write the equations $e_S(\mathbf{x}) = \varepsilon^1$ (for all $S \in A^1$) as,

$$\left\{ \begin{array}{l} v(12) - x_1 - x_2 = \frac{1}{3}, \\ v(13) - x_1 - x_3 = \frac{1}{3}, \\ v(23) - x_2 - x_3 = \frac{1}{3}, \end{array} \right. \qquad \text{or,} \qquad \Lambda^1 \mathbf{x} = \begin{bmatrix} \frac{14}{3} \\ \frac{17}{3} \\ \frac{23}{3} \end{bmatrix},$$

where

$$\Lambda^1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The reduction of Λ^1 to row-echelon form is as follows:

$$\Lambda^1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, the rank of Λ^1 is 3. According to Fromen's algorithm, the algorithm terminates with the nucleolus solution as $y = (y_1, y_2, y_3) = (1\frac{1}{3}, 3\frac{1}{3}, 4\frac{1}{3})$, which is the same as that found in

Example 1.

B.3.2 Application of Fromen's Algorithm [5] for the Calculation of Nucleolus Solution of a Cooperative Game with a Non-Empty Core

We use the cooperative example in Example 2 to illustrate Fromen's LP method for a cooperative game with a non-empty core.

1. Since the first linear problem in Fromen's Algorithm is the same as that in Maschler et al.'s algorithm, we learn from Appendix B.1.2 that $\varepsilon^1 = -1$ and

$$\begin{aligned} X^1 &= \{\mathbf{x}:\ x_1 \le 2,\ x_2 = 1,\ x_3 \le 4,\ x_1 + x_3 = 5\}, \\ A^1 &= \{\{2\}, \{1,3\}\}, \\ B^1 &= B^0 \setminus A^1 = \{\{1\}, \{3\}, \{1,2\}, \{2,3\}\}, \end{aligned}$$

The equations $e_S(\mathbf{x}) = \varepsilon^1$ (for all $S \in A^1$) can be thus written as,

$$\begin{cases} v(2) - x_2 = -1, \\ v(13) - x_1 - x_3 = -1, \end{cases} \qquad \text{or,} \qquad \Lambda^1 \mathbf{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

where

$$\Lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Since the rank of Λ^1 is obviously smaller than 3, we have to proceed with the second linear problem.

2. We define the set Z^1 as,

$$\begin{aligned} Z^1 &= \{\mathbf{x}: \textstyle\sum_{i=1}^{3} x_i = 6;\ x_i \ge v(i),\ i \in B^1;\ x_2 = 1,\ x_1 + x_3 = 5\} \\ &= \{\mathbf{x}: x_i \ge v(i),\ i \in B^1;\ x_2 = 1,\ x_1 + x_3 = 5\}; \end{aligned}$$

and develop the second linear problem as,

$$
\begin{aligned}
& \min\ z = \alpha_2' - \alpha_2'' \\
\text{subject to} & \\
\{1\}:\ & v(1) - x_1 \leq \alpha_2' - \alpha_2'', \\
\{3\}:\ & v(3) - x_3 \leq \alpha_2' - \alpha_2'', \\
\{1,2\}:\ & v(12) - (x_1 + x_2) \leq \alpha_2' - \alpha_2'', \\
\{2,3\}:\ & v(23) - (x_2 + x_3) \leq \alpha_2' - \alpha_2'', \\
\mathbf{x} \in X^1:\ & x_1 \geq v(1) = 0,\ x_3 \geq v(3) = 0, \\
& x_1 + x_2 \geq v(12) = 1,\ x_2 + x_3 \geq v(23) = 3, \\
S \in A^1:\ & x_2 = 1, \\
S \in A^1:\ & x_1 + x_3 = 5, \\
\alpha_1:\ & \alpha_1' \geq 0,\ \alpha_1'' \geq 0.
\end{aligned}
$$

Solving the above linear problem , we find that $\varepsilon^2 = -\frac{3}{2}$ and

$$
\begin{aligned}
X^2 &= \{\mathbf{x}:\ x_1 = \frac{3}{2},\ x_2 = 1,\ x_3 = \frac{7}{2}\}, \\
A^2 &= \{\{1\},\{1,2\},\{2,3\}\}, \\
B^2 &= B^1 \setminus A^2 = \{\{3\}\}.
\end{aligned}
$$

The equations $e_S(\mathbf{x}) = \varepsilon^i$ (for all $S \in A^i$, $i = 1,2$) can be thus written as,

$$
\begin{cases}
v(2) - x_2 = -1, \\
v(13) - x_1 - x_3 = -1, \\
v(1) - x_1 = -\frac{3}{2}, \\
v(12) - x_1 - x_2 = -\frac{3}{2}, \\
v(23) - x_2 - x_3 = -\frac{3}{2},
\end{cases}
\qquad \text{or,} \qquad \Lambda^1 \mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ \frac{3}{2} \\ \frac{5}{2} \\ \frac{9}{2} \end{bmatrix},
$$

where

$$
\Lambda^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.
$$

We use the Gauss elimination method to find the echelon form of the matrix Λ^2 as,

$$\Lambda^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which means that the rank of Λ^2 is equal to 3. Thus, the algorithm terminates with the nucleolus solution as $y = (y_1, y_2, y_3) = (\frac{3}{2}, 1, \frac{7}{2})$, which is the same as that found in Example 2.

Appendix C Proof of Theorem 1

For a three-player empty-core cooperative game in characteristic form, we find from (1) that $e_i(\mathbf{x}) \leq 0$, for $i = 1, 2, 3$. However, since the core of the game is empty, at least one of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ must be positive. Otherwise, if $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are all equal to or less than zero, then using (2) we have $v(12) \leq x_1 + x_2$, $v(13) \leq x_1 + x_3$ and $v(23) \leq x_2 + x_3$, which implies that the core is not empty.

Therefore, the maximal excess must be one of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. Accordingly, in order to minimize the maximal excess to find the nucleolus solution, we should change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to minimize the maximum of $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. If $e_{12}(\mathbf{x})$ is the maximum, then we reduce the value of x_3 and increase the values of x_1 and x_2; but, this raises the excesses $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. As a result, $e_{12}(\mathbf{x})$ must be equal to the maximum of $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$. For example, if $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) > e_{23}(\mathbf{x})$, we can then reduce the values of x_3 and x_2 but increase the value of x_1, in order to make both $e_{12}(\mathbf{x})$ and $e_{13}(\mathbf{x})$ smaller; but this increases the excess $e_{23}(\mathbf{x})$. Thus, the process terminates only when $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are equal. A similar argument applies to the case in which $e_{13}(\mathbf{x})$ or $e_{23}(\mathbf{x})$ is the maximum.

In conclusion, after we minimize the maximal excess, the excesses $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ must be equal and also, they must be nonnegative, i.e., $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x}) \geq 0$. We can then solve the following equations,

$$\begin{cases} v(12) - x_1 - x_2 = v(13) - x_1 - x_3, \\ v(12) - x_1 - x_2 = v(23) - x_2 - x_3, \\ v(123) = x_1 + x_2 + x_3, \end{cases}$$

and find the values of x_i, for $i = 1, 2, 3$. Because the payoffs of all three players have been chosen to minimize the maximal excess, we cannot make any change on the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to reduce the other excesses $e_i(\mathbf{x})$ $(i = 1, 2, 3)$. Otherwise, the maximal excess will be increased. Thus, the nucleolus $\mathbf{y} = (y_1, y_2, y_3)$ is found as (4). ■

Appendix D Proof of Lemma 1

We show the sufficiency and necessity of these conditions.

Sufficiency. In this part, if one of five conditions is satisfied, then the largest excesses are reduced to the minimum. We begin by showing the first sufficient condition. Since $x_1 = x_2 = x_3 = \frac{1}{3}v(123)$; $v(123) \geq 3v(12)$, $v(123) \geq 3v(13)$ and $v(123) \geq 3v(23)$, we use (1) and (2) to find that

$$\begin{aligned} e_1(\mathbf{x}) &= e_2(\mathbf{x}) = e_3(\mathbf{x}) = -\frac{1}{3}v(123), \\ e_{12}(\mathbf{x}) &= v(12) - v(123) + x_3 = v(12) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \\ e_{13}(\mathbf{x}) &= v(13) - v(123) + x_2 = v(13) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \\ e_{23}(\mathbf{x}) &= v(23) - v(123) + x_1 = v(23) - \frac{2}{3}v(123) \leq -\frac{1}{3}v(123), \end{aligned}$$

which implies that at least one of the excesses $e_i(\mathbf{x})$ (i = 1, 2, 3) is the largest. Next we prove that the largest excesses arrive to the minimum when $x_1 = x_2 = x_3 = v(123)/3$, that is, $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. Suppose that $e_1(\mathbf{x})$ is the largest excess and $e_2(\mathbf{x})$ and $e_3(\mathbf{x})$ are both less than $e_1(\mathbf{x})$. In order to decrease $e_1(\mathbf{x}) = -x_1$, we should increase the value of x_1. However, since $x_1 + x_2 + x_3 = v(123)$, we must reduce the value of x_2 and/or the value of x_3, thereby increasing the excess $e_2(\mathbf{x}) = -x_2$ and/or $e_3(\mathbf{x}) = -x_3$. This continues until $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. When either $e_2(\mathbf{x})$ or $e_3(\mathbf{x})$ is the largest, we can obtain the same result. Thus, we can conclude that if $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$, $v(123) \geq 3v(12)$, $v(123) \geq 3v(13)$ and $v(123) \geq 3v(23)$, then the largest excesses arrive to the minimum; thus we reach the first sufficient condition.

We then discuss the second sufficient condition. From (5) we have $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$. Recalling from (2) that $e_3(\mathbf{x}) = -x_3$ and $e_{12}(\mathbf{x}) = v(12) - v(123) + x_3$, we find that in order to reduce the excess $e_3(\mathbf{x})$, we should increase the value of x_3. However, this increases the value of $e_{12}(\mathbf{x})$. Therefore, we cannot change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ to reduce both $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$ simultaneously. Next, we show that $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$ are two largest excesses; that is, we should prove that $e_3(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, $e_3(\mathbf{x}) - e_2(\mathbf{x}) \geq 0$, $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0$ and $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0$.

1. From (1) we find that $e_3(\mathbf{x}) - e_1(\mathbf{x}) = -x_3 + x_1$. Using (5) we compute

$$e_3(\mathbf{x}) - e_1(\mathbf{x}) = \frac{v(123) + v(12)}{2} - x_2 - \frac{v(123) - v(12)}{2} = v(12) - x_2,$$

   and we find that $e_3(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, which results from (6).

2. From (1) we find that $e_3(\mathbf{x}) - e_2(\mathbf{x}) = -x_3 + x_2$. Using (5) we compute

$$e_3(\mathbf{x}) - e_2(\mathbf{x}) = x_2 - \frac{v(123) - v(12)}{2},$$

   and we find that $e_3(\mathbf{x}) - e_2(\mathbf{x}) \geq 0$ according to (6).

3. From (1) and (2) we find that $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) = -v(13) + v(123) - x_3 - x_2 = -v(13) + x_1$.

Using (5) we compute

$$e_3(\mathbf{x}) - e_{13}(\mathbf{x}) = -v(13) + x_1 = \frac{v(123) + v(12)}{2} - v(13) - x_2,$$

and we find that $e_3(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0$ according to (6).

4. From (1) and (2) we also find that $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) = -v(23) + v(123) - x_1 - x_3 = -v(23) + x_2$. Using (5) we compute $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) = -v(23) + x_2$ and, using (6), we find that $e_3(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0$.

Similarly, we can show the sufficient conditions 3 and 4. Next we discuss the last sufficient condition. Using (11) we have $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x}) = [v(12) + v(13) + v(23) - 2v(123)]/3$. Next, we show that these three excesses are the largest, i.e., $e_{12}(\mathbf{x}) \geq e_i(\mathbf{x})$, $i = 1, 2, 3$. From (1) and (2) we find that $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) = v(12) - v(123) + x_3 + x_1 = v(12) - x_2$. According to (11) we have $x_2 = [v(123) + v(12) + v(23) - 2v(13)]/3$, and thus compute $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) = [2v(12) + 2v(13) - v(123) - v(23)]/3$. From (12) we find that $e_{12}(\mathbf{x}) - e_1(\mathbf{x}) \geq 0$, or, $e_{12}(\mathbf{x}) \geq e_1(\mathbf{x})$. We can analogously show the $e_{12}(\mathbf{x}) \geq e_2(\mathbf{x})$ and $e_{12}(\mathbf{x}) \geq e_3(\mathbf{x})$. Hence, we conclude that if the conditions (11) and (12) are satisfied, then the largest excesses are reduced to the minimum.

Necessity. In this part, if the largest excesses are reduced to the minimum, then at least one of five conditions must be satisfied. Note that each of the six excesses $e_1(\mathbf{x})$, $e_2(\mathbf{x})$, $e_3(\mathbf{x})$, $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ could be largest. Next, assuming that each of these excesses is the largest, we change the imputation $\mathbf{x} = (x_1, x_2, x_3)$ under the constraint $x_1 + x_2 + x_3 = v(123)$ until it is reduced to the minimum.

1. If $e_1(\mathbf{x})$ is the largest excess, then according to (1) we can increase the value of x_1 to reduce this excess. However, from (2) we find that increasing x_1 shall raise the excess $e_{23}(\mathbf{x})$. Furthermore, because $x_1 + x_2 + x_3 = v(123)$, we should decrease x_2 and x_3, so increasing $e_2(\mathbf{x})$ and $e_3(\mathbf{x})$ in (1). Note that the excesses $e_{12}(\mathbf{x})$ and/or $e_{13}(\mathbf{x})$ in (2) decrease when we decrease x_2 and/or x_3 to reduce the largest excess $e_1(\mathbf{x})$. Thus, the largest excess reaches the minimum when $e_1(\mathbf{x}) = e_{23}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$.

   Consider the case that $e_1(\mathbf{x}) = e_{23}(\mathbf{x})$ and they are the largest excesses. Using (1) and (2) we have the equation $-x_1 = v(23) - v(123) + x_1$ and solve it to obtain $x_1 = [v(123) - v(23)]/2$. Since $x_1 + x_2 + x_3 = v(123)$, we reach (9). In addition, since $e_1(\mathbf{x})$ is the largest excess, we have

$$\begin{cases} e_1(\mathbf{x}) - e_2(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_3(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{12}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0, \end{cases} \quad \text{or} \quad \begin{cases} -x_1 + x_2 \geq 0, \\ -x_1 + x_3 \geq 0, \\ x_2 \geq v(12), \\ x_3 \geq v(13), \end{cases}$$

   which is equivalent to (10). Thus, the fourth condition including (9) and (10) corresponds to this case.

   Next, we discuss the case that $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$ and they are the largest excesses. According to (1), we find $-x_1 = -x_2 = -x_3$ and use $x_1 + x_2 + x_3 = v(123)$ to attain the imputation $\mathbf{x} = (x_1, x_2, x_3) = (v(123)/3, v(123)/3, v(123)/3)$. Because $e_1(\mathbf{x})$ is the largest excess, we have

$$\begin{cases} e_1(\mathbf{x}) - e_{12}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{13}(\mathbf{x}) \geq 0, \\ e_1(\mathbf{x}) - e_{23}(\mathbf{x}) \geq 0, \end{cases} \quad \text{or} \quad \begin{cases} x_2 \geq v(12), \\ x_3 \geq v(13), \\ x_1 \geq v(23). \end{cases}$$

Replacing x_i (for $i = 1, 2, 3$) with their solutions and simplifying the above inequalities give $v(123) \geq \max(3v(12), 3v(13), 3v(23))$. Thus, we reach the first necessary condition.

2. Similarly, if $e_2(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_2(\mathbf{x}) = e_{13}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. We can also analogously show that the third necessary condition including (7) and (8) corresponds to the case that $e_2(\mathbf{x}) = e_{13}(\mathbf{x})$.
3. Similarly, if $e_3(\mathbf{x})$ is the largest excess, then it arrives to the minimum when $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$ or $e_1(\mathbf{x}) = e_2(\mathbf{x}) = e_3(\mathbf{x})$. We can also show that the second necessary condition including (5) and (6) corresponds to the case that $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$.
4. If $e_{12}(\mathbf{x})$ is the largest excess, then according to (2) we can decrease the value of x_3 to reduce this excess. However, from (2) we find that decreasing x_3 shall raise the excess $e_3(\mathbf{x})$. Furthermore, since $x_1 + x_2 + x_3 = v(123)$, we should increase x_1 and x_2, so increasing $e_{23}(\mathbf{x})$ and $e_{13}(\mathbf{x})$ in (2). Note that the excesses $e_1(\mathbf{x})$ and/or $e_2(\mathbf{x})$ in (1) decrease when we increase x_2 and/or x_3 to reduce the largest excess $e_{12}(\mathbf{x})$. Thus, the largest excess reaches the minimum when $e_{12}(\mathbf{x}) = e_3(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$.
   We have shown that the second necessary condition corresponds to the case that $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$. Next we use (2) to solve $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$, and obtain (11). Since $e_{12}(\mathbf{x})$ is the largest excess, we have $e_{12}(\mathbf{x}) - e_i(\mathbf{x}) \geq 0$, for $i = 1, 2, 3$; and we use (11) to simplify these three inequalities and reach (12). Hence, the fifth necessary condition including (11) and (12) corresponds to the case that $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$.
5. Similarly, if $e_{13}(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_{13}(\mathbf{x}) = e_2(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$; the former corresponds to the third necessary condition and the latter corresponds to the fifth necessary condition.
6. Similarly, if $e_{23}(\mathbf{x})$ is the largest excess, then it reaches the minimum when $e_{23}(\mathbf{x}) = e_1(\mathbf{x})$ or $e_{12}(\mathbf{x}) = e_{13}(\mathbf{x}) = e_{23}(\mathbf{x})$; the former corresponds to the fourth necessary condition and the latter corresponds to the fifth necessary condition.

This proves the lemma. ■

Appendix E Proof of Theorem 2

We can easily find from Lemma 1 that if $v(123) \geq \max(3v(12), 3v(13), 3v(23))$, the excesses $e_i(\mathbf{x})$ (for $i = 1, 2, 3$) are the largest and thus the imputation $\mathbf{x} = (x_1, x_2, x_3) = (v(123)/3, v(123)/3, v(123)/3)$ when the largest excesses are reduced to the minimum. Since we have obtained the values of x_i, for $i = 1, 2, 3$, we cannot decrease any other excess. Hence, we arrive to Case 1 in Theorem 2.

Next, we consider the situation in which the largest excess is minimized because the second condition in Lemma 1 is satisfied. Under the condition, $e_3(\mathbf{x}) = e_{12}(\mathbf{x})$, $x_3 = [v(123) - v(12)] / 2$ and the value of x_2 is determined under the constraint (6). By using (6), we consider the following four cases in which we minimize the second largest excesses.

1. If $v(123) \geq v(12) + 2v(23)$, $v(123) \geq v(12) + 2v(13)$ and $v(123) \leq 3v(12)$, then $\{[v(123) + v(12)]/2 - v(13)\} \geq v(12) \geq [v(123) - v(12)]/2 \geq v(23)$, and we can reduce (6) to $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$ and we can easily show that

$$\max\{v(23), v(13)\} \leq [v(123) - v(12)]/2 \leq v(12). \tag{16}$$

   Next, we choose an appropriate value of x_2 to minimize the second largest excesses subject to $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$. Except for the largest excesses $e_3(\mathbf{x})$ and $e_{12}(\mathbf{x})$, the other excesses are computed as

$$\begin{aligned} e_1(\mathbf{x}) &= -x_1 = x_2 - \frac{v(123) + v(12)}{2}, & (17)\\ e_2(\mathbf{x}) &= -x_2, & (18)\\ e_{13}(\mathbf{x}) &= x_2 - v(123) + v(13), \\ e_{23}(\mathbf{x}) &= v(23) - v(123) + x_1 = v(23) - \frac{v(123) - v(12)}{2} - x_2. \end{aligned}$$

   Using (16) we have $e_1(\mathbf{x}) \geq e_{13}(\mathbf{x})$ and $e_2(\mathbf{x}) \geq e_{23}(\mathbf{x})$, which implies that $e_1(\mathbf{x})$ and/or $e_2(\mathbf{x})$ could be the second largest excess. From (17) and (18) we find that the second largest excesses are reduced to the minimum as $e_1(\mathbf{x}) = e_2(\mathbf{x})$, or, $x_2 = [v(123) + v(12)]/4$, which satisfies the constraint $[v(123) - v(12)]/2 \leq x_2 \leq v(12)$. Since $x_1 + x_2 + x_3 = v(123)$, we compute $x_1 = x_2 = [v(123) + v(12)]/4$. We notice that the other excesses (i.e., $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$) cannot be reduced because the imputation $\mathbf{x}$ has been determined; thus, the nucleolus solution is $y_1 = y_2 = [v(123) + v(12)]/4$ and $y_3 = [v(123) - v(12)]/2$, which corresponds to the second case (with $i, j = 1, 2$ and $i \neq j$, and $k = 3$) in Theorem 2.
2. If $v(123) \geq v(12) + 2v(23)$, $v(123) \leq v(12) + 2v(13)$ and $v(12) \geq v(13)$, then $v(12) \geq \{[v(123) + v(12)]/2 - v(13)\} \geq [v(123) - v(12)]/2 \geq v(23)$, and we can reduce (6) to $[v(123) - v(12)]/2 \leq x_2 \leq \{[v(123) + v(12)]/2 - v(13)\}$. Similar to the last case, we can show that under this condition the nucleolus solution is computed as $y = (y_1, y_2, y_3) = ([v(12) + v(13)]/2, [v(123) - v(13)]/2, [v(123) - v(12)]/2)$, which corresponds to the third case (with $i = 1$, $j = 2$ and $k = 3$) in Theorem 2.
3. If $v(123) \leq v(12) + 2v(23)$, $v(123) \geq v(12) + 2v(13)$ and $v(12) \geq v(23)$, then we find the formula of computing nucleolus solution for the third case (with $i = 2$, $j = 1$ and $k = 3$) in Theorem 2.
4. If $v(123) \leq v(12) + 2v(23)$, $v(123) \leq v(12) + 2v(13)$ and $v(123) + v(12) \geq 2[v(13) + v(23)]$, then we find the formula of computing nucleolus solution for the fourth case (with $i, j = 1, 2$ and $i \neq j$, and $k = 3$) in Theorem 2.

Similar to our above analysis, we can analyze the third and fourth conditions in Lemma 1, and reach the corresponding results in Theorem 2.

From Lemma 1 we find that under the condition (12), the excesses $e_{12}(\mathbf{x})$, $e_{13}(\mathbf{x})$ and $e_{23}(\mathbf{x})$ are the largest and the triple imputation x is obtained as (11). Thus, we cannot decrease any other excess. Hence, we arrive to fifth case in Theorem 2. ■

"**Analytic Solution for the Nucleolus of a Three-Player Cooperative Game**"

by M. Leng and M. Parlar

Submitted to *Naval Research Logistics* (NRL-09-0175)

Responses to the AE's Comments on the October 2009 Version

Thank you for your helpful comments on the October 2009 version of our paper. We read your report carefully and revised the paper in accordance with your comments and the comments provided by Reviewers 1 and 2[1]. We hope that you will find the revision satisfactory.

We summarize below our responses to your three minor comments:

1. **Comment 1**: "The authors need to be careful; about what they mean by "easy". This paper's intent is algebraic ease, not the ease of computational complexity. The Shapley value is NOT easy to compute as a size of the input (input being the set of players). For many games, it is NP-complete."

1. **Response:** We agree with your comment regarding the meaning of the word "easy." We have checked the whole technical note, and revised all relevant statements. For example, in our *previous* "Abstract" of the October 2009 version of our paper, we had stated "...This paper proposes an *easy-to-implement* algebraic method to compute the nucleolus solution analytically...". In our current "Abstract", we removed the words "*easy-to-implement*" and also shortened the abstract.

2. **Comment 2**: "Please check Case 6 & 7 on page 17. I get different expressions."
   Response: Thank you for your comment. We checked our formulas and the proofs of Lemma 1 and Theorem 2 **several times**. We find that the formulas for all cases are correct. In addition, we followed Reviewer 2's comments to remove our *previous* Table 3; we now have more compact formulas in current Theorem 2 on **page 6**.

3. **Comment 3**: "This paper can be significantly shortened. The sequential method and the example can be in an appendix. The main result and its proof in the main body."

   Response: We agree with you and the two reviewers who recommended to reduce the full paper to a short note. More specifically, we did our best to reduce our note's length by moving to online appendices the following materials:
   (a) In the October 2009 version of our paper, we provided an example in Section 1 (Introduction) to show how to transform a superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. This transformation is tangentially relevant to our note; thus, we moved this example to **online Appendix A** entitled "Transformation of a Superadditive and Essential Game to a "Zero-Normalized" Game."
   (b) As you and two reviewers suggested, we moved the sequential LP methods to **online Appendix B** entitled "Sequential LP Method for Computing the Nucleolus Solution."

[1] We identified the reviewer number from your report.

In addition, following your and Reviewer 1's comment, we used two other LP algorithms (i.e., Potters et al. [15] and Fromen [5]) to discuss the problems in two examples that we solved previously only by using Maschler et al.'s method [11] in the October 2009 version of our paper. Later, we provide more explanation on this issue.

(c) Since the three proofs for this note take up substantial amount of space, we decided to move them to **online Appendices C, D and E**.

In addition to moving the above materials to online appendices, we also did our best to shorten the main text of this note. As a result, **the note now includes only 9 pages**.

Below is a summary of our responses to the two reviewers' major comments that you mentioned in your report.

- **Responses to Reviewer 1's Major Comments that you mentioned**

  1. **Comment**: "The first referee questions the extent of the contribution and its fit to NRL and raises some issues similar to mine about the claims made about the wide use of the Nucleolus. Further this referee rightly points out that the computational comparisons are not fair as there are other recipes that are faster than the benchmark used by the authors. The authors should address the points made by this referee in any subsequent revision."

     Response: We agree with you and Reviewer 1 that, in addition to Maschler et al.'s algorithm, it is important to use other available LP methods to solve the two examples that we had solved by using Maschler et al.'s algorithm in the October 2009 version. In order to choose the methods that are faster than Maschler et al.'s, we compare all LP methods listed in Table 1. We find from our comparison that the LP approaches in Potters et al. [15] and Fromen [5] seem to be the two "relatively easy-to-implement" ones compared with other LP methods. For our detailed discussions on the LP methods in Table 1, see **the first four paragraphs in online Appendix B**.

     Accordingly, we first described Maschler et al.'s sequential LP approach in **online Appendix B.1** and presented two examples to illustrate this approach. Note that this material is almost the same as that in *previous* Section 2 of the October 2009 version. We still keep this material in **online Appendix B** because Maschler et al.'s LP approach in [11, 1979] is an *early* one in applying the LP method to the calculation of the nucleolus solution. Then, in **online Appendices B.2** and **B.3**, we respectively summarize the LP methods by Potters et al. [15, 1996] and Fromen [5, 1997], and illustrate these two methods with two numerical examples (that we had used to illustrate Maschler et al.'s LP approach).

     From our descriptions and illustrations of the three algorithms in the **online Appendix B**, we can find that the sequential LP method is not easy to use for the calculation of the nucleolus solution.

- **Responses to Reviewer 2's Major Comments that you mentioned**

1. **Comment**: "The second referee thinks this paper can be technical note. This referee raises 9 issues. The first one is easily resolved, the authors should just make a clearer discussion of their assumptions (they do this, but it needs to be more clear). Please see point 6 as well with respect to this. The rest of the comments are mainly editorial, please pay careful attention in any subsequent revision and respond to this referee. (9) raises a issue in the proof of Theorem 1. The proof is correct, please add a few lines explaining why this works."

   Response: We have very carefully considered all of the nine comments from Reviewer 2, and revised our paper accordingly .

   (a) **Reviewer 2's Comment 6**. We agree that we should have provided more compact results rather than many formulas in *previous* Table 3. Accordingly, we now summarize our previous formulas and present our results in **Theorem 2 on page 6**.

   Moreover, as Reviewer 2 suggested, it would be helpful to use the current compact results to obtain some insights. We used our results in current Theorem 2 to demonstrate that the nucleolus is not always monotonic; this has been proved by Megiddo [12]. For our discussion, see **page 7**, indicated by **AE.2.6**—short for **AE's Comment that is also Reviewer 2**'s **Comment 6** on the page margin.

   (b) **Reviewer 2's Comment 9**. Following your and Reviewer 2's comment, we now first prove that $e_{12}(x) = e_{23}(x) = e_{13}(x)$, and then show that, because the core is empty, $e_{12}(x) = e_{23}(x) = e_{13}(x) > 0$. For our new proof, see **online Appendix C**.

For this revision, we considered *all* comments from you and the two reviewers, and did our best to reduce our previous full paper to the current technical note. We hope that you will be satisfied with the note.

"**Analytic Solution for the Nucleolus of a Three-Player Cooperative Game**"

by M. Leng and M. Parlar

Submitted to *Naval Research Logistics* (NRL-09-0175)

Responses to Reviewer 1's Comments on the October 2009 Version[1]

Thank you for your helpful comments on the October 2009 version of our paper. We carefully read your report entitled "*Referee Report on 'Analytic Solution for the Nucleolus of a Three-Player Cooperative Game'*," and revised the paper in accordance with your comments and the comments provided by the AE and Reviewer 2. We hope that you will find the revision satisfactory.

We summarize below our responses to your comments:

- **Comment 1**: "**Length vs. Contribution**. I agree with the authors that the use of these analytical expressions saves time and has practical value when three-player games are analyzed. However, it is my belief that a tool that has such limited application ($n = 3$) does not merit a full-length paper in *Naval Research Logistics*. As I mentioned above, the results can help in calculating the nucleolus in small games, so I can potentially see it published as a short, $5 - 6$ pages technical note (for instance, I am not sure if the lengthy description of the sequential LP method for computing the nucleolus is necessary, especially in the main body of the paper)."

  Response: We agree with you, the AE and Reviewer 2's recommendation to reduce our previous full paper to a short technical note. More specifically, we did our best to reduce our note's length by moving to online appendices the following materials:

  1. In our previous full paper (i.e., the October 2009 version), we provided an example in Section 1 (Introduction) to show how to transform a superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. This transformation is only tangentially relevant to our note; thus, we moved this example to **online Appendix A** entitled "Transformation of a Superadditive and Essential Game to a 'Zero-Normalized' Game."
  2. As you, the AE and Reviewer 2 suggested, we moved the sequential LP methods to **online Appendix B** entitled "Sequential LP Method for Computing the Nucleolus Solution." In addition, following your comment, we used two other LP algorithms to solve the two examples that were solved only by using Maschler et al. [11, 1979] in the October 2009 version of our paper. For our detailed discussion on this issue, see our response to your next comment "**Comparisons with Other Methods**."
  3. Since the three proofs for this note are quite lengthy, we decided to move them to **online Appendices C, D and E**.

  In addition to moving the above materials to online appendices, we also did our best to shorten the main body of this note. As a result, **the note only includes 9 pages**.

[1] We identified your reviewer number from the AE's report.

- **Comment 2**: "**Comparisons with Other Methods**. While the analytical expressions from this work provide a time-saving tool, the authors have chosen to compare the "speed" of their method to the algorithm provided by Maschler et al. (1979), which, in view of their Table 1, is clearly dominated by several methods that were developed afterwards (and can work with more than 3 players). In the interest of fairness, it would be useful to look at how some of the other, faster methods perform on the examples provided."

  Response: In order to choose the methods that are faster than Maschler et al.'s, we compared all LP methods listed in Table 1. We find from our comparison that the LP approaches in Potters et al. [15, 1996] and Fromen [5, 1997] should be two "relatively easy-to-implement" ones compared with other LP methods. For our detailed discussions on the LP methods in Table 1, see **the first four paragraphs in online Appendix B**.
  Accordingly, we first described Maschler et al.'s sequential LP approach in **online Appendix B.1** and presented two examples to illustrate this approach. Note that this material is almost the same as that in *previous* Section 2 of the October 2009 version. We still keep this material in **online Appendix B** because Maschler et al.'s LP approach in [11, 1979] is an *early* one in applying the LP method to the calculation of the nucleolus solution. Then, in **online Appendices B.2** and **B.3**, we respectively summarize the LP methods by Potters et al. [15, 1996] and Fromen [5, 1997], and illustrate these two methods with two numerical examples (which we had used to illustrate Maschler et al.'s LP approach).
  From our descriptions and illustrations of three algorithms in **online Appendix B**, we can find that the sequential LP method is not easy to use for the calculation of the nucleolus solution.

- **Comment 3**: "**Usage of the Nucleolus**. Unrelated to the above comments, I am not sure if the paper with purely technical contribution is a good fit with *Naval Research Logistics*; I leave this decision to the AE. However, the authors mention in the Introduction (p.1) that "The nucleolus has become an important solution concept used wildly in cooperative games. . . " and that "The nucleolus solution concept has been widely used to solve a variety of problems for cooperative games in characteristics function form." I am not sure if such strong statements are justified, as complexities in calculation of the nucleolus seemed to have prohibited its widespread use in practical (more specifically, OM and/or logistics) applications. I would either like to see some additional evidence supporting these statements (the authors reference four papers, one of them their own), or the statements should be modified and toned down."

  Response: We agree with you that the computational complexity of the nucleolus solution restricts the applications of this concept, and we should modify our previous strong statements. This was also suggested by the AE. Accordingly, in the current technical note, we have deleted our *previous* statement "*The nucleolus has become an important solution concept used widely in cooperative games. . .*" For this revision, please see **page 1**, indicated by **R1.3.1**—short for **Reviewer 1's Comment 3** (**Question 1**) on the page margin.

In addition, we also agree with you that, in the October 2009 version, our *previous* statement "*The nucleolus solution concept has been widely used to solve a variety of problems for cooperative games in characteristics function form.*" was unduly strong.

We accordingly deleted the above statement, and instead wrote the following: "Nucleolus solution is an important concept in cooperative game theory even though it is not easy to calculate. As Maschler et al. [11, p. 336] pointed out, the nucleolus satisfies some desirable properties—e.g., it always exists uniquely in the core if the core is non-empty, and is therefore considered an important fair division scheme. As a consequence, some researchers have used this concept to analyze business and management problems; but, due to the complexity of calculations, the nucleolus has not been extensively used to solve allocation-related problems.**"** For the statements, see **page 1**, indicated by **R1.3.2** on the page margin.

In addition to the above two statements that you mentioned, we carefully checked the complete technical note, and revised some a few other "strong" statements. We hope that you will find our current statements satisfactory.

- <u>**Comment 4**</u>: "**Cost-Saving Agreements and Allocation Rules**. On a related note, one of the papers referenced in support of the use of the nucleolus is Barton (1992). The authors quote this work in providing arguments that the nucleolus is a good tool for allocation of joint costs among entities who share a common resource, as the use of the nucleolus can reduce the possibility that one or more entities may wish to withdraw from resource-sharing arrangement. However, in the interest of fairness and objectivity, the authors should then also quote Megiddo (1974), who showed that the nucleolus is not monotonic in the aggregate. This implies that, for instance, if a cost overrun occurs, some entities may benefit by having their share of costs reduced (see, e.g., Young 1985 for an example). Thus, some other solution concepts (e.g., the Shapley value, which may not be in the core but satisfies this and other monotonicity criteria , or the per capita nucleolus (Grotte 1970), which is in the core) may be better candidates for cost-sharing agreements."

  <u>**Response**</u>: Thank you for suggesting three new, and important, references. We agree that if, in Barton's cost allocation problem, the cost for running the common resource increases, then the nucleolus solution may suggest a lower cost allocated to some entities. This is possible because, as Megiddo [12] proved, the nucleolus is not always monotonic. In fact, it has been showed that some other concepts satisfy the monotonicity property and may be used instead of the nucleolus. For example, Young [23] proved that the Shapley value is a unique, monotonic solution, even though this concept may not be in the core if the core is non-empty. In [8], Grotte normalized the nucleolus—by dividing the "excess" of each coalition by the number of players in the coalition—and correspondingly, introduced the new concept "*per capita* (*normalized*) nucleolus" as an alternative one of the nucleolus solution. Grotte showed that the per capita nucleolus is monotonic and also always exists in the core if the core is non-empty. Thus, for some cost-sharing problems such as that in Barton [1], the per capita nucleolus may be better than the nucleolus solution; but, we notice that the calculation for

the per capita nucleolus could be more complicated than that for the nucleolus.

Following your comment, we include the above material in our technical note; see **page 2**, indicated by **R1.4** on the page margin.

For this revision, we considered *all* comments from you, the AE and Reviewer 2, and did our best to reduce our previous full paper to the current technical note. We hope that you will be satisfied with the note.

"**Analytic Solution for the Nucleolus of a Three-Player Cooperative Game**"

by M. Leng and M. Parlar

Submitted to *Naval Research Logistics* (NRL-09-0175)

Responses to Reviewer 2's Comments on the October 2009 Version[1]

Thank you for your helpful comments on the October 2009 version of our paper. We carefully read your report entitled "*Review for NRL-09-0175: Analytic Solution for the Nucleolus of a Three-Player Cooperative Game*", and revised the paper in accordance with your comments and the comments provided by the AE and Reviewer 1. We hope that you will find the revision satisfactory.

As you, the AE and Reviewer 1 suggested, we reduced our previous full paper to a short technical note. Following your detailed comments, we shortened our previous paper and now present a short note that includes only 9 pages. Several related materials (the sequential LP methods, proofs, etc.) that are supplementary to our technical note are provided in **online Appendices A–E.**

We summarize below our responses to your nine comments:

1. **Comment 1**: "The paper does not explicitly state that the nucleolus is always unique. Instead at various places, the authors state/show that the closed-form expressions they provide result in a unique solution. I find this somewhat confusing. Instead it should be stated early on (probably in the Introduction section where certain properties of the nucleolus are introduced) that the nucleolus is unique. Then later on in the paper there is no need to state that the closed-form expressions result in a unique solution. Since they are claimed to characterize the nucleolus, the solution has to be unique."

   Response: We agree that the uniqueness of the nucleolus solution should have been mentioned in Section 1 (Introduction) only. Following your comment, we stated in Section 1 (Introduction) that the nucleolus solution is unique. For our statement, see **page 1**, indicated by **R2.1**—short for **Reviewer 2's Comment 1** on the page margin. In other sections of our technical note, we don't mention the uniqueness again.

2. **Comment 2**: "I suggest that the last sentence of the first paragraph on page 3 (the sentence that starts with even though) be removed. It is not clear how/if the method proposed in this paper can be generalized to problems with more than three players."

   Response: We agree with you, and have removed the sentence. See **page 3**, indicated by **R2.2** on the page margin.

3. **Comment 3**: "Towards the middle of page 6 (around line 24), the authors state the nucleolus solution of a game as found by the analytical approach proposed in the paper. However at this point, the reader does not know anything about the analytical approach and hence it may be better not to mention the nucleolus solution at all at this point."

   Response: We agree with you; now, this material has been moved to **online Appendix A**. We did this because of the following reason: In our previous full paper (i.e., the October

[1] We identified your reviewer number from the AE's report.

2009 version), we provided an example in Section 1 (Introduction) to show how to transform a superadditive, and essential three-player game to a "*0-normalized*" game with zero characteristic values of all one-player coalitions. This example includes our statement that you questioned. Since the transformation is only tangentially relevant to our note, we moved this example (including that statement) to **online Appendix A** entitled "Transformation of a Superadditive and Essential Game to a 'Zero-Normalized' Game."

Since our algebraic approach is presented in the technical note, we think that it should be acceptable to mention the calculation of the nucleolus in **online Appendix A**.

4. **Comment 4**: "I am confused by the first two paragraphs of section 2. The section starts by stating that Maschler et al. developed an LP-based algorithm for computing the nucleolus solution of a cooperative game. Then references are made to alternative solution techniques proposed by Wang, Owen, Barton and Carter and Walker. The way I interpret this paragraph is that Wang and Owen fail to provide guidance on finding the nucleolus solution when the LPs in question have alternative solutions. Barton and then Carter and Walker tried to resolve this problem, but their approaches were not entirely accurate and the current paper refines the method proposed by Barton and Carter. Is my interpretation correct? If my interpretation is correct, where does the method proposed by Maschler et al. stand in this discussion? Is their method silent on how to interpret alternative solutions, too? From reading the first paragraph of section 2, I get the impression that Maschler et al. already proposed a way to interpret alternative solutions in computing the nucleolus solution and this paper is merely providing two examples. Is that the case? If so, what is the contribution of section 2?"

   Response: We believe the first two paragraphs in our previous Section 2 of the October 2009 version were not clear enough and thus confused you. In fact, as an early publication regarding the sequential LP method, Maschler et al. [11] used the concept of lexicographic centre to develop an LP procedure involving $O(4^n)$ minimization LP problems. This LP approach has been adopted by some textbooks (e.g., Wang [22]) as a "typical" method to calculate the nucleolus solution. However, neither in Maschler et al. [11] nor in the textbooks that applied Maschler et al.'s algorithm to some examples, we could find any detailed (i.e., *step by step*) explanations about the LP method in [11], especially when a linear problem exhibits alternative optimal solutions. Therefore, in *previous* Section 2 of the October 2009 version, we wrote the first two paragraphs just to show that previous Section 2 was needed in the October 2009 version of our paper.

   As the AE and Reviewer 1 suggested, the sequential LP method is not very important to our technical note; and it should be instead moved to an online appendix. Accordingly, we moved our LP discussions to **online Appendix B**. Thus, we reduced the first two paragraphs in our previous Section 2 of the October 2009 version to a short paragraph; see **the first paragraph in online Appendix B.1**.

   You may notice that, in **online Appendix B,** we also used a few other LP methods to solve the two examples that we had solved by using Maschler et al.'s algorithm in the October

2009 version of our paper. We did this because, as Reviewer 1 suggested, we were asked to choose the methods that are faster than Maschler et al.'s algorithm and compare them with Maschler et al.'s. We found that the LP approaches in Potters et al. [15] and Fromen [5] were two "relatively easy-to-implement" ones compared to the other LP methods.

Accordingly, in **online Appendix B**, we first described Maschler et al.'s sequential LP approach in **Section B.1** and presented two examples to illustrate this approach. Note that this material is almost the same as that in *previous* Section 2 of the October 2009 version. We still keep this material in **online Appendix B** because Maschler et al.'s LP approach in [11] is an *early* one in applying the LP method to the calculation of the nucleolus solution. Then, in **Sections B.2** and **B.3**, we respectively summarize the LP methods by Potters et al. [15] and Fromen [5], and illustrate these two methods with two numerical examples (that we had used to illustrate Maschler et al.'s LP approach).

5. **Comment 5**: "Right after Theorem 2, two examples are provided to demonstrate that the algebraic method proposed in this paper computes a unique nucleolus solution. But as I stated before, the nucleolus solution is unique and so, if the method proposed is accurate, it should compute a unique solution. Therefore I suggest that these two examples are removed from the paper."

   Response: We agree with you that these two examples should be deleted. In the current technical note, we **don't** provide any examples after Theorem 2 to show the uniqueness.

6. **Comment 6**: "Using the fact that some of the formulas in Table 3 have a common structure (for example, cases 2, 6, and 10 share the same structure both in terms of conditions and the resulting formulas. Similarly 5, 9 and 13 have the same structure.), it should be possible to collapse Table 3. This more compact representation will both shorten the paper and also may allow the authors to generate some insights based on the closed form expressions they obtain. For example, can anything be said about the monotonicity properties of the nucleolus solution (beyond what is known for the nucleolus solution in general, e.g. the fact that the nucleolus payoff to a player may decrease even when $v(N)$ increases)?

   Response: Thank you for this helpful comment; we agree with you that we it would be useful to provide more compact results rather than several formulas in *previous* Table 3. Accordingly, we summarized our previous formulas and presented our results in **Theorem 2 on page 6**.

   Moreover, as you suggested, we also used our results in current Theorem 2 to demonstrate that the nucleolus is not always monotonic; this has been proved by Megiddo [12]. For our discussion, see **page 7**, indicated by **R2.6** on the page margin.

7. **Comment 7**: "The Maple worksheets that are available from the authors' web site are potentially useful in computing the nucleolus solution algebraically. However I don't think Table 4 adds any value to the paper. Simply listing the nucleolus solutions obtained by the LP

and algebraic methods on the same table (not surprisingly both methods come up with the same solutions) does not demonstrate the computational convenience of the method proposed (On a different note, I don't think showing the computational simplicity of the proposed method over the LP method would make a strong contribution anyway). In addition, since the algebraic method is proven to be exact, stating that it finds the same solutions as the LP method for a variety of games from the literature is not useful either. Hence I suggest this table be removed from the paper."

Response: We agree with you, and accordingly removed previous Table 4. This table disappears in our current technical note.

8. **Comment 8**: "I suggest that the last paragraph of the Conclusions section be removed. Since it is not clear how the method can be extended to more than three players (more importantly to a general n-player game), it is not really useful to list it as a potential extension of the current paper. In addition, I also suggest that the last two sentences of the second to last paragraph be removed, too. Again, it is not surprising that the algebraic method assures the uniqueness of the nucleolus solution. It is also not fair to compare a computational method with close-form expressions in terms of how simple they are to use in computing the nucleolus solution."

   Response: We have already removed the last two sentences of our *previous* second paragraph and also the last paragraph in the October 2009 version of our paper.

9. **Comment 9**: "I have a question regarding the proof of Theorem 1. Since the core of the game is empty, we know that at least one of the excesses $e_{ij}(x)$ must be positive. But then further in the proof it is assumed that (if statement used) $e_{12}(x) = e_{23}(x) = e_{13}(x) > 0$. The closed-form expressions are derived under this assumption. Shouldn't the authors first argue that in the nucleolus solution $e_{12}(x) = e_{23}(x) = e_{13}(x)$ (the way it is argued in the proof of Lemma 1) and then state that, because the core is empty, it must be that $e_{12}(x) = e_{23}(x) = e_{13}(x) > 0$?"

   Response: Following your comment, we now first prove that $e_{12}(x) = e_{23}(x) = e_{13}(x)$, and then show that, because the core is empty, $e_{12}(x) = e_{23}(x) = e_{13}(x) > 0$. For our new proof, see **online Appendix C**.

For this revision, we considered *all* comments from you, the AE and Reviewer 1, and did our best to reduce our previous full paper to the current technical note. We hope that you will find the note satisfactory.