Note: You are welcome to bring this formula sheet to the exam, but it is your responsibility to check the accuracy of the formulas given here.

Chapter 24: Review of Probability

Conditional probability:	$\Pr(E_2 \mid E_1) = \dfrac{\Pr(E_1 \cap E_2)}{\Pr(E_1)}$, if $\Pr(E_1) \neq 0$
Exponential density:	$f(t) = \lambda e^{-\lambda t}$, $t \geq 0$ with $E(T) = 1/\lambda$ and $\mathrm{Var}(T) = 1/\lambda^2$ $F(t) = \Pr(T \leq t) = 1 - e^{-\lambda t}$
Poisson density:	$\Pr(N(t) = n) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0, 1, \ldots$ with $E[N(t)] = \lambda t$
Uniform density:	$f(x) = \dfrac{1}{b-a}$, for $a \leq x \leq b$
Expected value:	$E(X) = \sum_k x_k \Pr(X = x_k)$, or $E(X) = \int_0^\infty x f(x)\, dx$.
Variance:	$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
Expectations by conditioning:	$E(X) = \sum_y E(X \mid Y = y) \Pr(Y = y)$ $E(X) = \int_0^\infty E(X \mid Y = y) g(y)\, dy$
Probabilities by conditioning:	$\Pr(A) = \sum_y \Pr(A \mid Y = y) \Pr(Y = y)$ $\Pr(A) = \int_0^\infty \Pr(A \mid Y = y) g(y)\, dy$

Chapter 16: Markov Chains (Skip)

- n-step transition matrices: $\mathbf{P}^{(n)} = \mathbf{P}^n$. Unconditional probabilities are $\mathbf{\Pi}^{(n)} = \mathbf{\Pi}^{(n-1)}\mathbf{P}$, for $n = 1, 2, \ldots$ where $\mathbf{\Pi} = (\pi_0, \pi_1, \ldots, \pi_M)$.
- Steady-state equations: $\pi_j = \sum_{i=0}^M \pi_i p_{ij}$ and $\sum_{j=0}^M \pi_j = 1$; or in matrix-vector notation, $\mathbf{\Pi} = \mathbf{\Pi}\mathbf{P}$ and $\mathbf{\Pi}\mathbf{e} = 1$, where $\mathbf{e}' = (1, 1, \ldots, 1)$.
- Cost calculation for complex functions: Calculate using conditional probabilites from

$$E(\text{Cost}) = \sum_{j=0}^{M} E(\text{Cost} \mid X = j) \Pr(X = j).$$

- Mean first passage times: $\mu_{ij} = 1 + \sum_{k \neq j} p_{ik}\mu_{kj}$. (Note that $\mu_{ii} = 1/\pi_i$.)

- Absorbing chains:

$$\begin{aligned}\mathbf{P} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}, \quad \mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}, \\ \mathbf{NR} &= \text{Pr(End up in an absorbing state} \mid \text{start elsewhere)}\end{aligned}$$

- Cramer's rule:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- CTMC: Balance equation is from the principle of "Rate Out = Rate In."

Chapter 17: Queueing Theory

Birth-Death Processes:

$$\begin{array}{rrcl} & \text{Rate Out} & = & \text{Rate In} \\ n = 0: & P_0\lambda_0 & = & P_1\mu_1 \\ n = 1: & (\lambda_1 + \mu_1)P_1 & = & \lambda_0 P_0 + \mu_2 P_2 \\ n = 2: & (\lambda_2 + \mu_2)P_2 & = & \lambda_1 P_1 + \mu_3 P_3 \\ \text{Any } n \geq 1: & (\lambda_n + \mu_n)P_n & = & \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}. \end{array}$$

Solution to the above is obtained from

$$P_n = P_0 \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n}, \quad n = 1, 2, \ldots \quad \text{and} \quad \sum_{n=0}^{\infty} P_n = 1.$$

Operating Characteristics of Queues

- In all models, $\bar{\lambda}/(s\mu)$ is the utilization factor, i.e., the expected fraction time the individual server(s) are busy, where $\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n$.

- We use the following shorthand notation in this formula sheet.

Shorthand Notation	**Explanation**
$M/M/1$	Poisson input, exponential service time model with 1 server
$M/M/s$	Poisson input, exponential service time model with s servers
$M/M/1/K$	The $M/M/1$ model with finite queue
$M/M/s/K$	The $M/M/s$ model with finite queue
$M/M/1/\cdot/N$	The $M/M/1$ model with finite population
$M/M/s/\cdot/N$	The $M/M/s$ model with finite population
$M/G/1$	Poisson input, general service time model with 1 server
$M/E_k/1$	Poisson input, Erlang service time model with 1 server

	P_0	P_n
$M/M/1$	$1-\rho$	$\rho^n P_0,\ n \geq 1$
$M/M/s$	$\frac{1}{\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s![1-\lambda/(s\mu)]}}$	$\begin{cases} \frac{(\lambda/\mu)^n P_0}{n!},\ n = 1, 2, \ldots, s \\ \frac{(\lambda/\mu)^n P_0}{s! s^{n-s}},\ n = s, s+1, \ldots \end{cases}$
$M/M/1/K$	$\begin{cases} \frac{1-\rho}{1-\rho^{K+1}}, & \lambda \neq \mu \\ \frac{1}{K+1} & \lambda = \mu \end{cases}$	$\begin{cases} \rho^n P_0, & \lambda \neq \mu \\ \frac{1}{K+1}, & \lambda = \mu \end{cases},\ n = 1, \ldots, K$
$M/M/s/K$	$\frac{1}{\sum_{n=0}^{s} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \sum_{n=s+1}^{K} \left(\frac{\lambda}{s\mu}\right)^{n-s}}$	$\begin{cases} \frac{(\lambda/\mu)^n P_0}{n!}, & 1 \leq n \leq s \\ \frac{(\lambda/\mu)^n P_0}{s! s^{n-s}}, & s \leq n \leq K \end{cases}$
$M/M/1/\cdot/N$	$\frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!} \left(\frac{\lambda}{\mu}\right)^n}$	$\frac{N!}{(N-n)!} \left(\frac{\lambda}{\mu}\right)^n P_0,\ n = 1, 2, \ldots, N$
$M/M/s/\cdot/N$	$\frac{1}{\sum_{n=0}^{s-1} \frac{N!}{(N-n)!n!} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{N} \frac{N!}{(N-n)!s!s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n}$	$\begin{cases} \frac{N!}{(N-n)!n!} \left(\frac{\lambda}{\mu}\right)^n P_0, & 1 \leq n \leq s \\ \frac{N!}{(N-n)!s!s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0, & s \leq n \leq N \end{cases}$

	L	L_q
$M/M/1$	$\dfrac{\lambda}{\mu-\lambda}$	$\dfrac{\lambda^2}{\mu(\mu-\lambda)}$
$M/M/s$	$L_q + \dfrac{\lambda}{\mu}$	$\frac{P_0(\lambda/\mu)^s\rho}{s!(1-\rho)^2}$
$M/M/1/K$	$\begin{cases} \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}, & \lambda \neq \mu \\ \frac{K}{2}, & \lambda = \mu \end{cases}$	$L-(1-P_0)$
$M/M/s/K$	$\sum_{n=0}^{s-1} nP_n + L_q + s\left(1-\sum_{n=0}^{s-1} P_n\right)$	$\begin{cases} \frac{P_0(\lambda/\mu)^s\rho[1-\rho^{K-s}-(K-s)\rho^{K-s}(1-\rho)]}{s!(1-\rho)^2}, & \lambda \neq s\mu \\ \frac{1}{2}\frac{P_0(\lambda/\mu)^s[K-s+(K-s)^2]}{s!}, & \lambda = s\mu \end{cases}$
$M/M/1/\cdot/N$	$N - \frac{\mu}{\lambda}(1-P_0)$	$N - \frac{\lambda+\mu}{\lambda}(1-P_0)$
$M/M/s/\cdot/N$	$\sum_{n=0}^{s-1} nP_n + L_q + s\left(1-\sum_{n=0}^{s-1} P_n\right)$	$\sum_{n=s}^{N}(n-s)P_n$
$M/G/1$	$\rho + L_q$	$\frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$
$M/E_k/1$	$\rho + L_q$	$\frac{1+k}{2k}\frac{\lambda^2}{\mu(\mu-\lambda)}$

	W	W_q	$\bar{\lambda}$
$M/M/1$	$\frac{1}{\mu-\lambda}$	$\frac{\lambda}{\mu(\mu-\lambda)}$	λ
$M/M/s$	$W_q + \frac{1}{\mu}$	L_q/λ	λ
$M/M/1/K$	$L/\bar{\lambda}$	$L_q/\bar{\lambda}$	$\lambda(1-P_K)$
$M/M/s/K$	$L/\bar{\lambda}$	$L_q/\bar{\lambda}$	$\lambda(1-P_K)$
$M/M/1/\cdot/N$	$L/\bar{\lambda}$	$L_q/\bar{\lambda}$	$\lambda(N-L)$
$M/M/s/\cdot/N$	$L/\bar{\lambda}$	$L_q/\bar{\lambda}$	$\lambda(N-L)$
$M/G/1$	$W_q + \frac{1}{\mu}$	$\frac{L_q}{\lambda}$	λ
$M/E_k/1$	$W_q + \frac{1}{\mu}$	$\frac{1+k}{2k}\frac{\lambda}{\mu(\mu-\lambda)}$	λ

	$\Pr(\mathcal{W} > t)$	$\Pr(\mathcal{W}_q > t)$	$\Pr(\mathcal{W}_q = 0)$
$M/M/1$	$e^{-\mu(1-\rho)t}$, $t \geq 0$	$\rho e^{-\mu(1-\rho)t}$, $t \geq 0$	$1-\rho$
$M/M/s$	$e^{-\mu t}\left[1 + \frac{P_0(\lambda/\mu)^s}{s!(1-\rho)} \cdot \frac{1-e^{-\mu t(s-1-\lambda/\mu)}}{s-1-\lambda/\mu}\right]$	$[1-\Pr(\mathcal{W}=0)]e^{-s\mu(1-\rho)t}$	$\sum_{n=0}^{s-1} P_n$

- $E(TC) = E(SC) + E(WC)$ where $E(SC) = C_s s$ and $E(WC) = C_w L$.

Chapter 26: The Application of Queueing Theory (Skip)

As before, $E(TC) = E(SC) + E(WC)$, but now,

- with $g(N)$ form, $E(WC) = \sum_{n=0}^{\infty} g(n)P_n$,
- with $h(\mathcal{W})$ form, $E(WC) = \lambda \int_0^{\infty} h(w) f_{\mathcal{W}}(w)\, dw$.