2011-10-24-EC01

Monday, October 24, 2011

6:13 PM

M/M/1 M/M/s

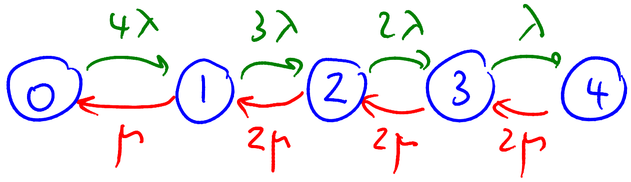
M/M/1/K M/M/s/K

→ M/M/1/·/N

f) M/M/s/·/N Finite Pop $(s>1)$

Same as (e) except s servers

Ex. Buses $N=4$, but $s=2$



Result

$$P_n = \begin{cases} \dfrac{N!}{n!\,(N-n)!}\left(\dfrac{\lambda}{\mu}\right)^n P_0, & n=0,1,\dots,s \\[2ex] \dfrac{N!}{(N-n)!\,s!\,s^{n-s}}\left(\dfrac{\lambda}{\mu}\right)^n P_0, & n=s,\dots,N \end{cases}$$

$$P_0 = \frac{1}{\displaystyle\sum_{n=0}^{s-1} \frac{N!}{n!\,(N-n)!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{N} \frac{N!}{(N-n)!\,s!\,s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n}$$

$$L_q = \sum_{n=s}^{N} (n-s)P_n$$

$$L = \sum_{n=0}^{s-1} nP_n + L_q + s\left(1-\sum_{n=0}^{s-1} P_n\right),$$

$\bar{\lambda} = \lambda(N-L)$ as before

$$W = L/\bar{\lambda}, \quad W_q = L_q/\bar{\lambda}$$

Comparison $s=1$, vs $s=2$ $(\lambda=10, \mu=20)$

Comparison $s=1$, vs $S=2$ $(\lambda=10,\ \mu=20)$

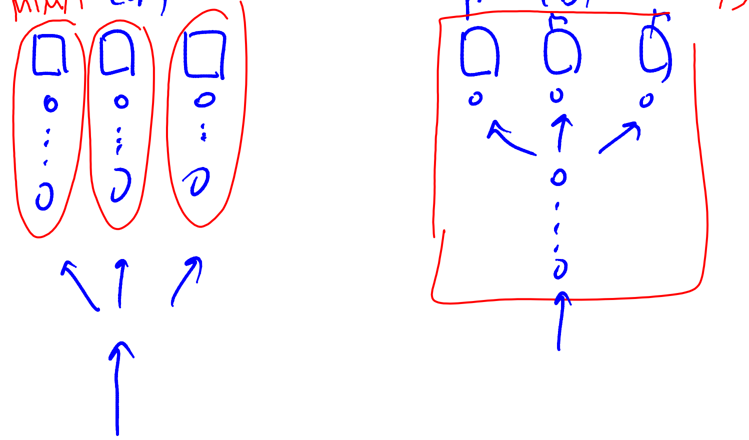
time = wk	M/M/1/./4	M/M/2/./4
L	2.19	1.47
L_q	1.28	0.20
W	0.12	0.05
W_q	0.07	0.008
P_0	0.09	0.18
$\bar{\lambda}$	18.69	25.28

$\bar{\lambda} = \lambda(N-L)$

Ex. Comparison of s parallel M/M/1 queues
vs. 1 M/M/S

$S=3$

M/M/1 (a) $\qquad$ μ (b) M/M/3

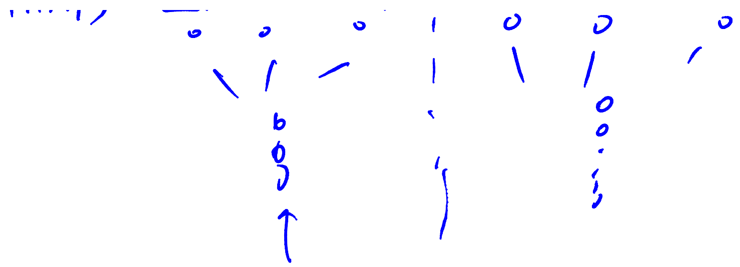


$\lambda = 100$ cust/day
$\mu = 50$ cust/day
Each M/M/1
$\lambda' = \frac{\lambda}{3} = 33.3$
$\rho = \frac{\lambda'}{\mu} = \frac{33.3}{50} = 0.66$

$\lambda = 100$ cust/day
$S = 3$
$\mu = 50$ cust/day
$\rho = \frac{\lambda}{s\mu} = \frac{100}{3 \cdot 50} = .66$

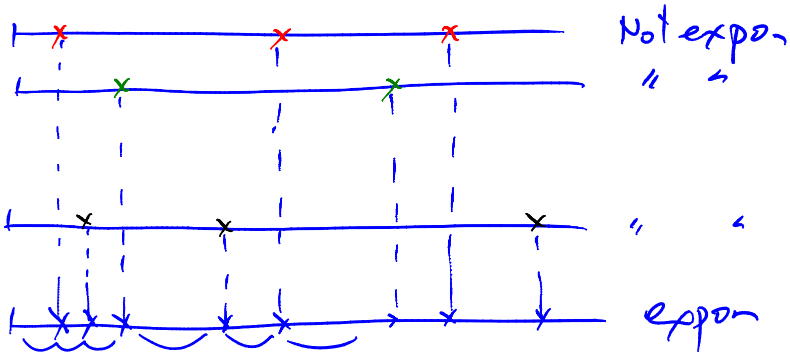
Option	$L_{subsystem}$	L	$L_{q\,(subsyst)}$	L_q	W	W_q
a	2	6	1.33	4.00	0.06	0.04
b	—	2.89	—	0.88	0.02	0.008

Fortino's
M/M/3



Ex. Pr. 17-6.32

See web



5. M/G/1 Queue + Its variations

Arrival: Poisson with rate λ (as before)

Service: General with mean $\frac{1}{\mu}$

variance σ^2

Two variations

M/D/1 : (constant service time) (robot, vending mach)

M/E_k/1 : Erlang

Erlang(k) : Sum of k individual expon's

each mean $\frac{1}{k\mu}$

" var $\frac{1}{(k\mu)^2}$

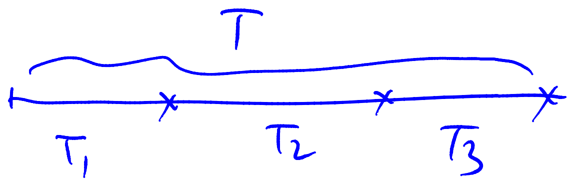
〃 var $\frac{1}{(k\mu)^2}$

$$T = T_1 + T_2 + \cdots + T_k$$

Mean $\quad \frac{1}{k\mu} + \frac{1}{k\mu} + \cdots + \frac{1}{k\mu} = \frac{1}{\mu}$: mean

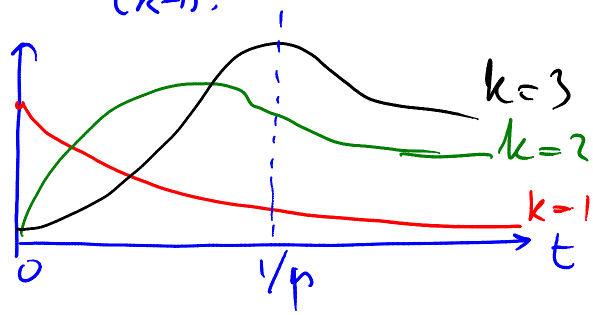
Var $\quad \frac{1}{(k\mu)^2} + \cdots + \frac{1}{(k\mu)^2} = k\frac{1}{(k\mu)^2} = \frac{1}{k\mu^2}$: variance

Ex.



$T = T_1 + T_2 + T_3$

$$f(t) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} e^{-k\mu t}, \quad t \geq 0$$



T is Erlang(3), $k=3$

$E(T) = 60$ min $= \frac{1}{\mu} \Rightarrow \mu = \frac{1}{60}$ /min

$Var(T) = \frac{1}{k\mu^2} = \frac{1}{3(\frac{1}{60})^2} = 1200$

Pollaczek-Khintchine

	P_0	L_q	L	W_q	W
M/G/1	$1-\rho$	$\frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$	$\rho + L_q$	L_q/λ	$W_q + \frac{1}{\mu}$
M/D/1	〃	ρ^2	〃	〃	〃

	P_0	L_q	L	W_q	W
M/G/1	$1-\rho$	$\frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$	$\rho+L_q$	L_q/λ	$W_q+\frac{1}{\mu}$
M/D/1	"	$\frac{\rho^2}{2(1-\rho)}$	"	"	"
M/E_k/1	"	$\frac{1+k}{2k}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)}$	"	$\frac{1+k}{2k}\cdot\frac{\lambda}{\mu(\mu-\lambda)}$	"
M/M/1	"	$\frac{\lambda^2}{\mu(\mu-\lambda)}=\frac{\rho^2}{1-\rho}$	"	$\frac{\lambda}{\mu(\mu-\lambda)}$	"

Ex. Comparison of M/D/1, M/M/1 + M/G/1

$\lambda = 5$ cust/hr, $\rho=\frac{\lambda}{\mu}=.63<1$ ✓
$\mu = 8$ cust/hr,

	M/D/1 ($\sigma^2=0$)	M/M/1 ($\sigma^2=\frac{1}{64}$)	M/G/1 ($\sigma^2=\frac{1}{16}$)
L	1.15	1.67	3.28
L_q	0.52	1.04	2.65
W	0.23	0.33	0.66
W_q	0.10	0.21	0.53

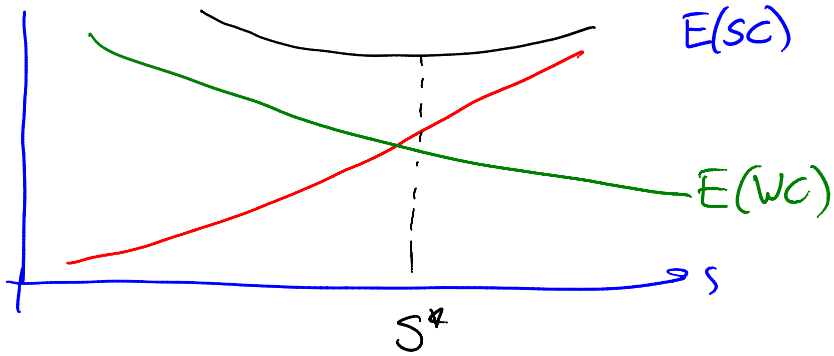
Other models
- Queueing networks
- Priority queues

6. Optimization in Queueing
(Ch.26 in appendix)

(Ch. 26 in appendix)

Decision variables: $\underbrace{s, \mu, \lambda}$

$\circ s \quad s, \mu \quad s, \lambda \quad \mu$



Unit (dimensional) analysis

$$TC = SC + WC$$

$$\frac{(\$)}{(\text{time})} = \frac{(\$)}{(\text{time})} + \frac{(\$)}{(\text{time})}$$

Consider $\quad s$: # servers [server] $\rightarrow$ service cost

$\qquad L$: # customers [cust] $\rightarrow$ wait cost

· we pay \$ per server per time (e.g. \$30/cashier-hr)

let $\quad C_s$: marginal cost of a server per unit time $\cdot \dfrac{[\$]}{[\text{serv}][\text{time}]}$

$\therefore \quad \boxed{E(SC) = s\,C_s}$ $\qquad \cdot \; \cancel{[\text{serv}]} \dfrac{[\$]}{\cancel{[\text{serv}]}(\text{time})} = \dfrac{[\$]}{(\text{time})}$

· Each customer spending an hour results in a cost (e.g., \$5/cust-hr)

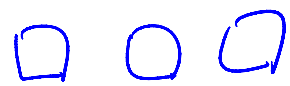
So, C_w: waiting cost per unit time for each cust : $\dfrac{[\$]}{(\text{cust})(\text{time})}$

$\therefore$ $\boxed{E(WC) = C_w L}$ $\quad \therefore \dfrac{[\$]}{\cancel{(\text{cust})}(\text{time})} \cdot \cancel{(\text{cust})} = \dfrac{(\$)}{(\text{time})}$

$$E(TC) = \underbrace{sC_s}_{E(SC)} + \underbrace{C_w L}_{E(WC)}$$

Ex. Pr. 17-10.3 (p. 826)

Currently 3 copy machines



$S=3$

Complaints about delays

Get more?

2000 hrs of work/yr $\left(= 40 \dfrac{\text{hr}}{\cancel{\text{wk}}} \times 50 \cancel{\text{wks}}\right) = 2000\,\text{hr}$

$\lambda = 30$ empl/hr
$1/\mu = 5$ min/empl.
$\mu = 12$ empl/hr

$$\rho = \frac{\lambda}{s\mu} = \frac{30}{3 \times 12} = \frac{30}{36} = .833$$

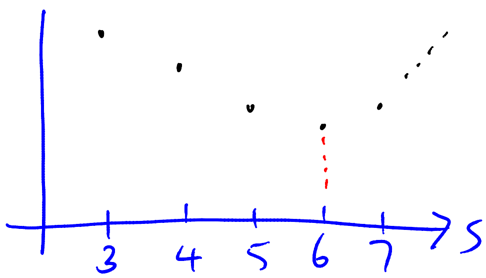
Lost prod. cost $\quad C_w = 25 \dfrac{[\$]}{(\text{empl})(\text{hr})}$

Rental cost of mach. \$3000/yr $\rightarrow \dfrac{3000}{2000} \overset{\$}{=} 1.5/\text{hr}$

$\underline{\$}$

$$\frac{\overline{yr}}{\frac{hr}{yr}} = \frac{\$}{hr}$$

$$C_s = 1.5 \; \frac{[\$]}{[mach](hr)}$$



$S^* = 6$

S	$C_s S$	$C_w L$	E(TC)
3	4.5	150	154.5
4	6	75	81.
5	7.5	65	73.5
6	9	63	72. ← Best!
7	10.5	62	72.5