2011-10-17-EC01

Monday, October 17, 2011
6:36 PM

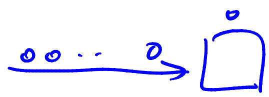
4. Exponential (Memoryless) Queueing Models (based on B&D processes)

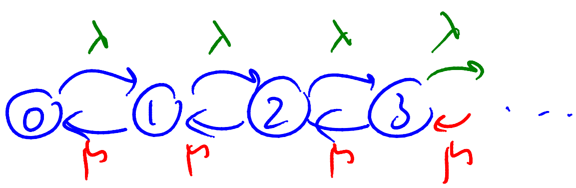
Capacity

Pop'n (Source)	$=\infty$	$<\infty$
$=\infty$	M/M/1 M/M/s	M/M/1/K M/M/s/K
$<\infty$	M/M/1/·/N M/M/s/·/N	Machine repair

usually $N = K$

a) M/M/1 System





$$C_n = \left(\frac{\lambda}{\mu}\right)^n = \rho^n, \quad n = 0, 1, 2, \ldots$$

rho

$$0 < \rho = \frac{\lambda}{\mu} < 1$$

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$$

$$P_2 = \qquad = \rho^2 P_0$$

$\vdots$

$$P_n = \qquad = \rho^n P_0$$

$\vdots$

Since $P_0 + P_1 + P_2 + \cdots = 1$

$$P_0 (1 + \rho + \rho^2 + \cdots) = 1$$

$$P_0 \frac{1}{1-\rho} = 1 \Rightarrow P_0 = 1 - \rho$$

So, $$P_n = \rho^n (1-\rho), \quad n = 0, 1, \ldots, \quad 0 < \rho < 1$$

Operating characteristics

L : Avg. # in system
L_q : " " " queue
L_s : " " " service

W : " time a cust. spends in system
W_q " " " " " " queue
W_s " " " " " service

L: (system)

n	0	1	2	3	$\cdots$
p_n	p_0	p_1	p_2	p_3	$\cdots$

$$L = \sum_{n=0}^{\infty} n p_n = \frac{\lambda}{\mu - \lambda} \quad \leftarrow \text{PhD guys Show!}$$

L_q (queue)

# in system n	0	1	2	3	$\cdots$
# in queue	0	0	1	2	$\cdots$
prob p_n	p_0	p_1	p_2	p_3	$\cdots$

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n = 0 \cdot p_0 + 0 \cdot p_1 + 1 \cdot p_2 + 2 \cdot p_3 + \cdots$$

$$= L - \rho = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad \leftarrow \text{PhD (Show!)}$$

$L_s = \rho$

~~Additional results on waiting time~~

Ex. Checkout desk at Library

$\lambda = 100$ cust/hr
$\mu = 120$ " $\left(\frac{1}{\mu} = 0.5 \text{ min/cust}\right)$

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$\rho = \frac{\lambda}{\mu} = 0.83 < 1$ ✓

$P_0 = 1 - \rho = .17$ (idle 17% time)

$L = \frac{\lambda}{\mu - \lambda} = 5$ cust

$L_q = 4.17$ "

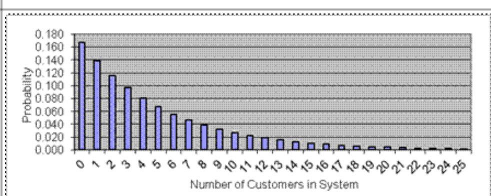
$L_s = \quad = 0.83$

<u>Little's formula</u>

$L = \lambda W$

$L_q = \lambda W_q$

$L_s = \lambda W_s$

Template for the M/M/s Queueing Model							
		Data				**Results**	
	$\lambda =$	100	(mean arrival rate)		L =	5.000	
	$\mu =$	120	(mean service rate)		$L_q =$	4.167	
	s =	1	(# servers)				
					W =	0.050	
	Pr(W > t) =	0.367879			$W_q =$	0.042	
	when t =	0.05					
					$\rho =$	0.833	
	Prob(W_q > t) =	0.306566					
	when t =	0.05			n	P_n	
					0	0.167	
					1	0.139	
					2	0.116	

					3	0.096	
					4	0.080	
					5	0.067	
					6	0.056	
					7	0.047	
					8	0.039	
					9	0.032	
					10	0.027	
					11	0.022	
					12	0.019	
					13	0.016	
					14	0.013	

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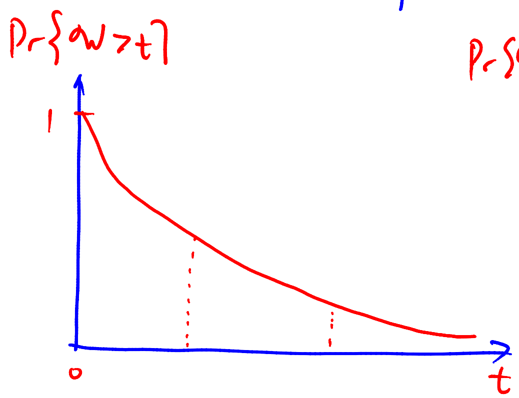
$$W = L/\lambda$$

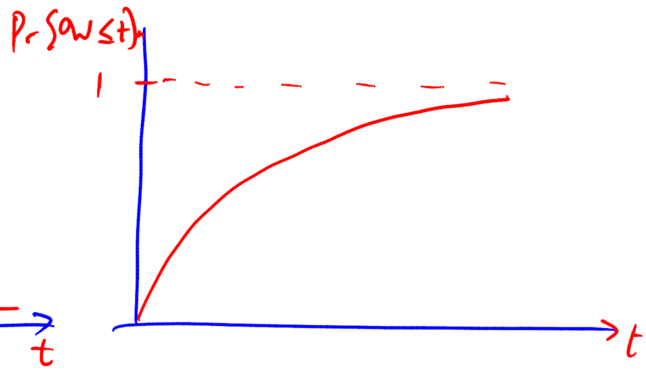
Additional result on w

$$\Pr\{w > t\} = e^{-\mu(1-\rho)t} \quad \leftarrow \text{PhD}$$

$$\Pr\{w \le t\} = 1 - e^{-\mu(1-\rho)t} \quad \checkmark$$

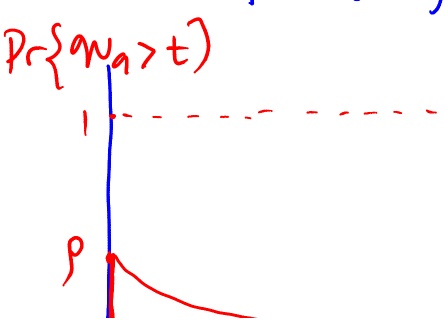
$$W = E(w) = \frac{1}{\mu-\lambda} \qquad \left(W = \frac{L}{\lambda} = \frac{\frac{\lambda}{\mu-\lambda}}{\lambda} = \frac{1}{\mu-\lambda} \right)$$

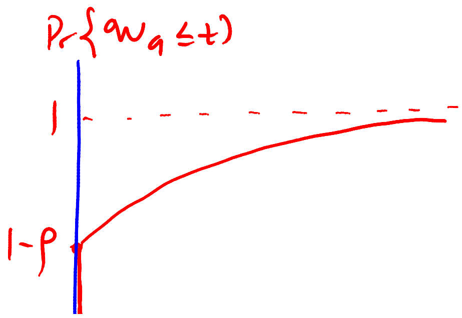


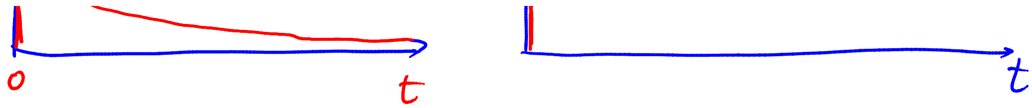


Additional result on w_q

$$\Pr\{w_q > t\} = \rho e^{-\mu(1-\rho)t} \quad \leftarrow \text{PhD}$$







$$W_q = E(W_q) = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$\left(\text{Little's.} \quad W_q = \frac{L_q}{\lambda} = \frac{\frac{\lambda^2}{\mu(\mu-\lambda)}}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}\right)$$

Ex. Library (Cont'd)

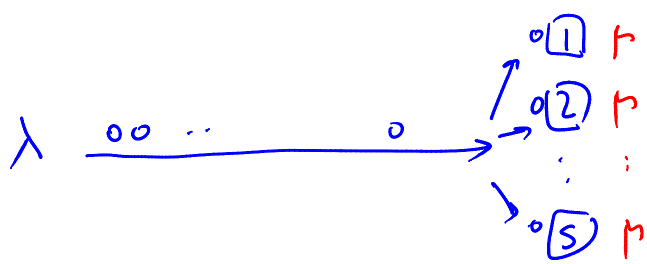
$$W = \frac{1}{\mu-\lambda} = \frac{1}{120-100} = 3 \text{ min}$$

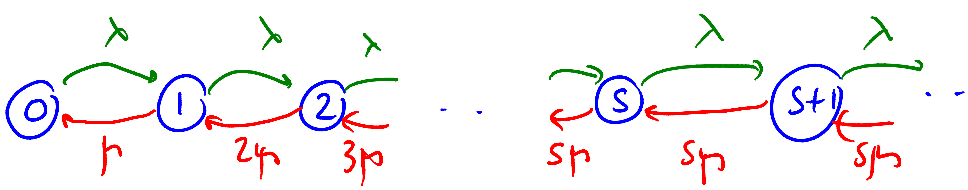
$$W_q = 2.5 \text{ min}$$

$$Pr\left\{W > \frac{3}{60}\right\} = \qquad = 0.368$$

$$= Pr(\text{wait} > 3 \text{ min})$$

b) M/M/s with $s \geq 1$ servers





$$\rho = \frac{\lambda}{s\mu} < 1$$

$$P_0 = \frac{1}{\left[\sum^{s-1} (\lambda/\mu)^n\right] . \ (\lambda/\mu)^s \quad \frac{1}{\qquad}}$$

$$\left(\sum_{n=0} \frac{\quad}{n!}\right) + \frac{\quad}{s!} \cdot \frac{\quad}{1-\frac{\lambda}{s\mu}}$$

$$p_n = \frac{(\lambda/\mu)^n p_0}{n!}, \quad n = 1, \dots, s$$

$$p_n = \frac{(\lambda/\mu)^n p_0}{s!\, s^{n-s}}, \quad n = s, s+1, \dots$$

So,

$$L_q = \frac{p_0 (\lambda/\mu)^s \rho}{s!\,(1-\rho)^2} \qquad W = W_q + \frac{1}{\mu}$$

$$W_q = \frac{L_q}{\lambda} \qquad L = L_q + \frac{\lambda}{\mu}$$

$$P_r\{\mathcal{W} > t\} = e^{-\mu t}\left[1 + \frac{p_0 (\lambda/\mu)^s}{s!\,(1-\rho)}\left(\frac{1 - e^{-\mu t(s-1-\lambda/\mu)}}{s-1-\lambda/\mu}\right)\right]$$

Ex. Library $s=2$

$\lambda = 100$
$\mu = 120$

	$s=1$	$s=2$
$\rho = \frac{\lambda}{s\mu}$	.83	0.416
p_0	.16	.41
L_q	4.16	.17
L	5	1.0
W_q	.04	.001
W	.05	.01

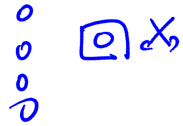
ρ = E(fraction of time an individual server busy)

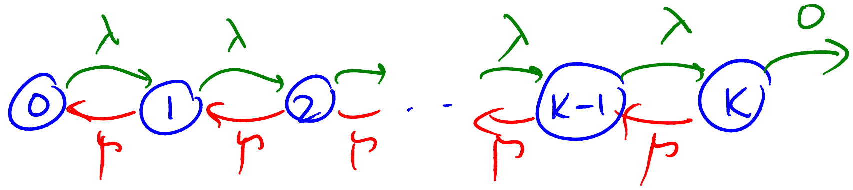
c) M/M/1/K system (Finite Queue)
↑ system capacity

(∞ source finite capacity)

Total in system is at most K

Ex. One barber





Case 1: $\lambda \neq \mu$, $\rho = \frac{\lambda}{\mu} \neq 1$

$$\left.\begin{aligned} p_1 &= \rho\, p_0 \\ p_2 &= \rho^2 p_0 \\ &\vdots \\ p_K &= \rho^K p_0 \end{aligned}\right\} \text{ from B\&D}$$

$$p_0 + p_1 + \cdots + p_K = 1 \Rightarrow p_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

$$\to \boxed{p_n = \frac{1-\rho}{1-\rho^{K+1}}\rho^n, \quad n = 0, 1, \ldots, K}$$

$$\to L = \sum_{n=0}^{K} n p_n = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}$$

$$\to L_q = L - (1 - p_0)$$

Is it true/false: $L = \lambda W$? False

Need $\bar{\lambda}$: effective arrival rate

Each cust λ — If K, then leave: λp_K leave

$\searrow$ If $< K$, " enter ∴ $\lambda - \lambda P_K = \lambda(1-P_K)$ actually enter

$$\bar{\lambda} = \lambda(1-P_K)$$

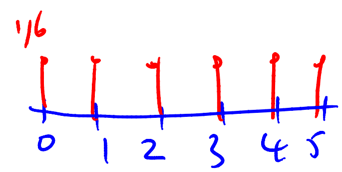
$$\Rightarrow \quad W = \frac{L}{\bar{\lambda}}, \quad W_q = \frac{L_q}{\bar{\lambda}} \quad //$$

<u>Case 2</u>: $\lambda = \mu$, $\rho = \frac{\lambda}{\mu} = 1$

$$P_0 = \frac{1}{K+1}$$

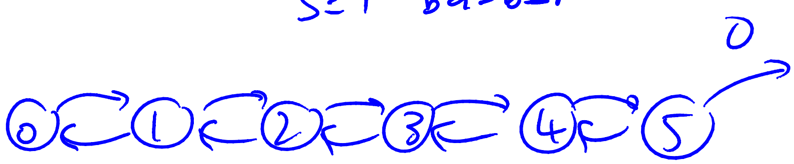
$K = 5$

and $P_n = \frac{1}{K+1}$, $n = 0, 1, \ldots, K$



$$L = \frac{1}{2} K$$

Ex. Barber $K = 5$ chairs
$s = 1$ barber



$\lambda = 16$ cust/hr
$1/\mu = 15$ min/cust
$\mu = 4$ cust/hr

$\rho = \frac{\lambda}{\mu} = 4$, $P_5 = .75$

$\bar{\lambda} = \lambda(1-P_5) = 16(1-.75) = 4$ cust/hr

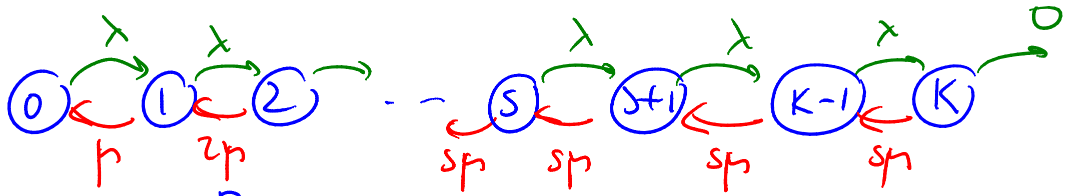
16 < 4 in, 12 turn away

$L = 4.67$ $\quad L_q = 3.66$

$W = 1.17$ hr $\quad W_q = 0.91$

$(S \leq K)$

d) $M/M/s/K$ $(s \geq 1)$ $(s \leq K)$



$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n P_0}{n!}, & n = 1, 2, \ldots, s \\[2ex] \dfrac{(\lambda/\mu)^n P_0}{s!\, s^{n-s}}, & n = s+1, \ldots, K \end{cases}$$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{s} \dfrac{(\lambda/\mu)^n}{n!}\right] + \dfrac{(\lambda/\mu)^s}{s!} \sum_{n=s+1}^{K} \left(\dfrac{\lambda}{s\mu}\right)^{n-s}}$$

Case 1 $\lambda \neq s\mu$ $(\rho \neq 1)$

$$L_q = \frac{P_0 (\lambda/\mu)^s \rho}{s!\,(1-\rho)^2}\left[1 - \rho^{K-s} - (K-s)\rho^{K-s}(1-\rho)\right], \quad \rho = \frac{\lambda}{s\mu}$$

$$L = \sum_{n=0}^{s-1} n P_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

$$W = \frac{L}{\bar{\lambda}}, \quad W_q = \frac{L_q}{\bar{\lambda}}, \quad \bar{\lambda} = \lambda(1 - P_K)$$

Case 2 $\lambda = s\mu$ $\rho = \dfrac{\lambda}{s\mu} = 1$

$$\lim_{\rho \to 1} L_q = \frac{1}{2}\, \frac{P_0 (\lambda/\mu)^s \left[K - s + (K-s)^2\right]}{s!}$$ (not in the book)

$$L = \sum_{n=0}^{s-1} n P_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

$$W = \frac{L}{\bar{\lambda}}, \quad W_q = \frac{L_q}{\bar{\lambda}}, \quad \bar{\lambda} = \lambda(1 - P_K)$$

$$W = \frac{L}{\bar{\lambda}}, \quad W_q = \frac{L_q}{\bar{\lambda}}, \quad \bar{\lambda} = \lambda(1 - P_K)$$

Ex. Two barbers

$K = 6$

$s = 2$

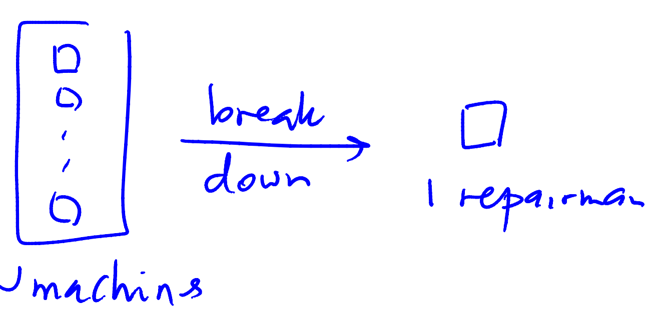
$K - 2 = 4$ wait

$\lambda = 16, \ \mu = 4, \ s = 2, \ \rho = \frac{\lambda}{s\mu} = 2$

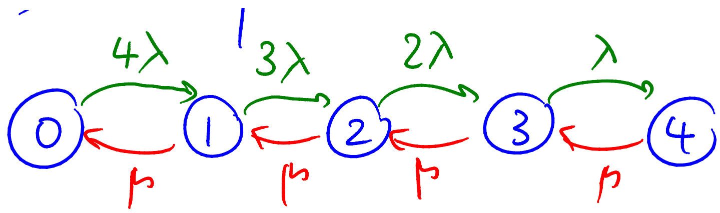
$P_6 = .51, \quad \bar{\lambda} = \lambda(1 - P_6) = 16(1 - .51) = \cancel{8.16}$

	$s=1$	$s=2$
L	4.67	5.07
L_q	3.66	3.09
W	1.17	0.64
W_q	0.91	0.39

e) $M/M/1/\cdot/N$: finite population



Ex. $N = 4, \ S = 1$ | state: # in repair
buses



<u>Results</u>

$$P_0 = \frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n}$$

$$P_n = \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 , \quad n = 1, \dots, N$$

$$L_q = N - \frac{\lambda+\mu}{\lambda}(1-P_0)$$

$$L = N - \frac{\mu}{\lambda}(1-P_0)$$

$$W = \frac{L}{\bar{\lambda}} , \quad W_q = \frac{L_q}{\bar{\lambda}}$$

What's $\bar{\lambda}$?

Each bus $\longrightarrow \lambda$

State	# Good	Arr. Rate (λ_n)	Prob
0	4	4λ	P_0
1	3	3λ	P_1
2	2	2λ	P_2
3	1	λ	P_3
4	0	0	P_4

$$\therefore \quad \bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = \sum_{n=0}^{N} (N-n)\lambda P_n = \lambda(N-L)$$

Ex. $\lambda = 10$ bus/wk , $N = 4$
$\mu = 20$ bus/wk , $S = 1$

$\Rightarrow \quad L = 2.19 \quad W = 0.12 \quad P_0 = 0.09$

$\Rightarrow$ $L = 2.19$ $W = 0.12$ $P_0 = 0.09$

$L_q = 1.28$ $W_q = 0.07$ $\bar{\lambda} = 18.09$

Utilization $= \dfrac{\bar{\lambda}}{\mu} = 0.9$