2011-10-03-EC01

Monday, October 03, 2011
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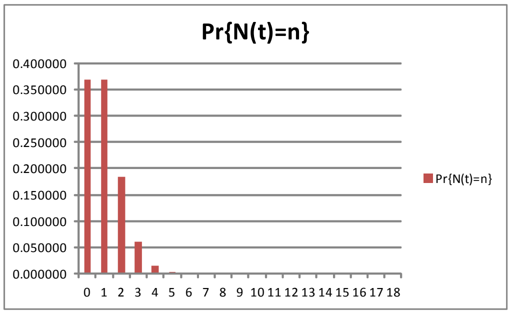
http://www.business.mcmaster.ca/courses/O711/ChapterComments/documents/ExponentialWorksheet.xlsx

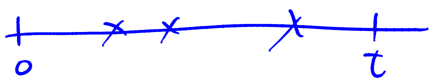


Poisson

Prop. 4. Exponential & Poisson relationship

Let $N(t)$: #arrivals in $(0,t)$: Counts

Ex.



$N(t)=3$

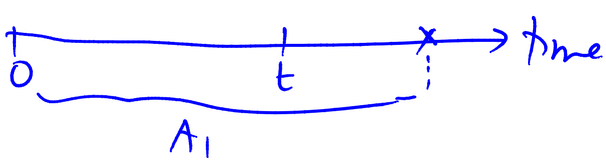
Suppose $N(t)$ is Poisson with rate α, i.e.,

$$\Pr[N(t)=n] = \frac{e^{-\alpha t}(\alpha t)^n}{n!}, \quad n=0,1,2,\ldots$$

Fact: Interarrivals are exponential with rate α
$\Leftrightarrow$ $N(t)$ is Poisson with parameter α

i.e.,
$$\underbrace{f(t)=\alpha e^{-\alpha t}}_{\text{time}} \Leftrightarrow \Pr(N(t)=n) = \underbrace{\frac{e^{-\alpha t}(\alpha t)^n}{n!}}_{\text{count}}$$

Ex: $n=0$ arrivals in $(0,t)$



$\Pr[N(t)=0] = e^{-\alpha t}$, \ Same (true for $n \geq 1$)

$Pr[N(t)=0] = e^{\cdots}$

$Pr[A_1 > t] = e^{-\alpha t}$) same (true for $n \geq 1$)

short interval

Prop. 5 $\quad Pr\{t < T \leq t+\Delta t \mid T > t\} \cong \alpha \Delta t$

PhD guys: Prove!

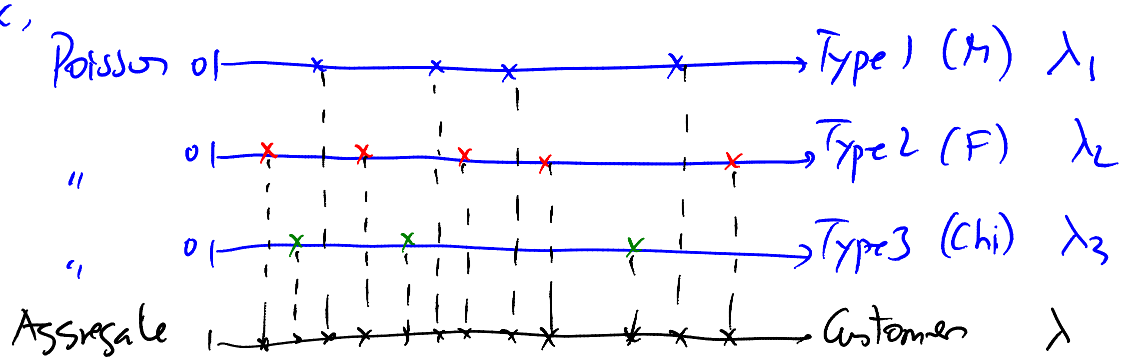
Ex. $\alpha = 5$ cust/hr

$\Delta t = 1$ min $= \frac{1}{60}$ hr

$\alpha \cdot \Delta t = 5 \cdot \frac{1}{60} = \frac{1}{12} = .083$

Prop. 6 Unaffected by "aggregation" or "disaggregation"

Ex.



Cust. process is also Poisson with rate

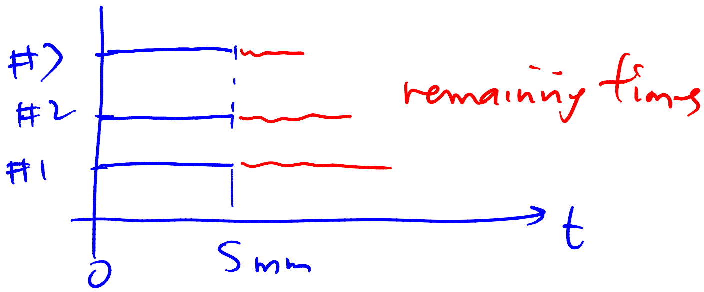
$\lambda = \lambda_1 + \lambda_2 + \lambda_3$

Ex. Similar to Pb. 17.4.5 (p. 815) $\quad f(t) = \mu e^{-\mu t}$, $E(T) = \frac{1}{\mu}$

Servers	1	2	3	
Mean $\frac{1}{\mu}$	20	15	10	mins
Time	T_1	T_2	T_3	expon

rate	$1/20$	$1/15$	$1/10$
	$\exp(1/20)$	$\exp(1/15)$	$\exp(1/10)$

Each server has been busy with the current customer for 5 mins already



Find E(remaining time until next service completion)?

Memorylessness $\Rightarrow$ remaining time for each is still exp. (Prop. 2)

So, $U = \min(T_1, T_2, T_3)$

$T_1 \sim \exp\left(\frac{1}{20}\right)$ rate

$T_2 \sim \exp\left(\frac{1}{15}\right)$

$T_3 \sim \exp\left(\frac{1}{10}\right)$

rate

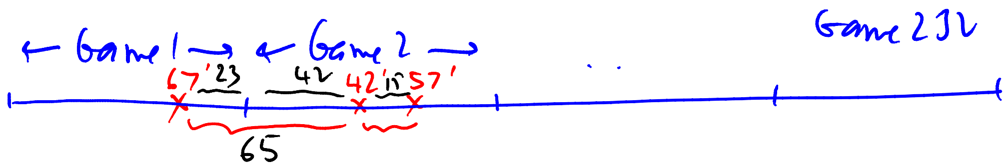
Prop. 3 $\Rightarrow$ $U \sim \exp\left(\frac{1}{20} + \frac{1}{15} + \frac{1}{10}\right) = \exp\left(\frac{13}{60}\right)$

So, $E(U) = \frac{60}{13} = 4.61$ min

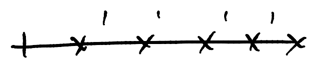
Ex. World Cup

2002	Korea-Japan	64	games
1998	France	64	"
1994	USA	52	

1990 Italy $\dfrac{52}{232}$ s



Cameroon: 1 Romania: 2
Arg = 0 Sov. U : 0
67' 42'
57'



574 intergoal durations → 575 goals

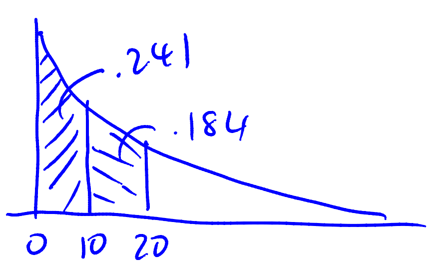
Intergoal Duration	Actual	Empirical Prob	Theoret Prob	Expect
0 – 10	144	.250	.241	138
10 – 20	106	.184	.182	105
⋮		⋮	⋮	⋮
120 - 130	6	.010	.008	5
> 130	16	.028	.028	16
	574			

Arrival rate: $\lambda = \dfrac{575}{232} = 2.47 \dfrac{\text{goals}}{\text{90-min game}}$

$E(\text{time between goals}) = \dfrac{1}{\lambda} = \dfrac{1}{2.47} = 0.40 \dfrac{\text{90-min game}}{\text{goals}}$

$\Rightarrow 0.40 \times 90 = 36 \text{ min}$

χ^2 (chi-Square) test ⇒ exp. is a good fit

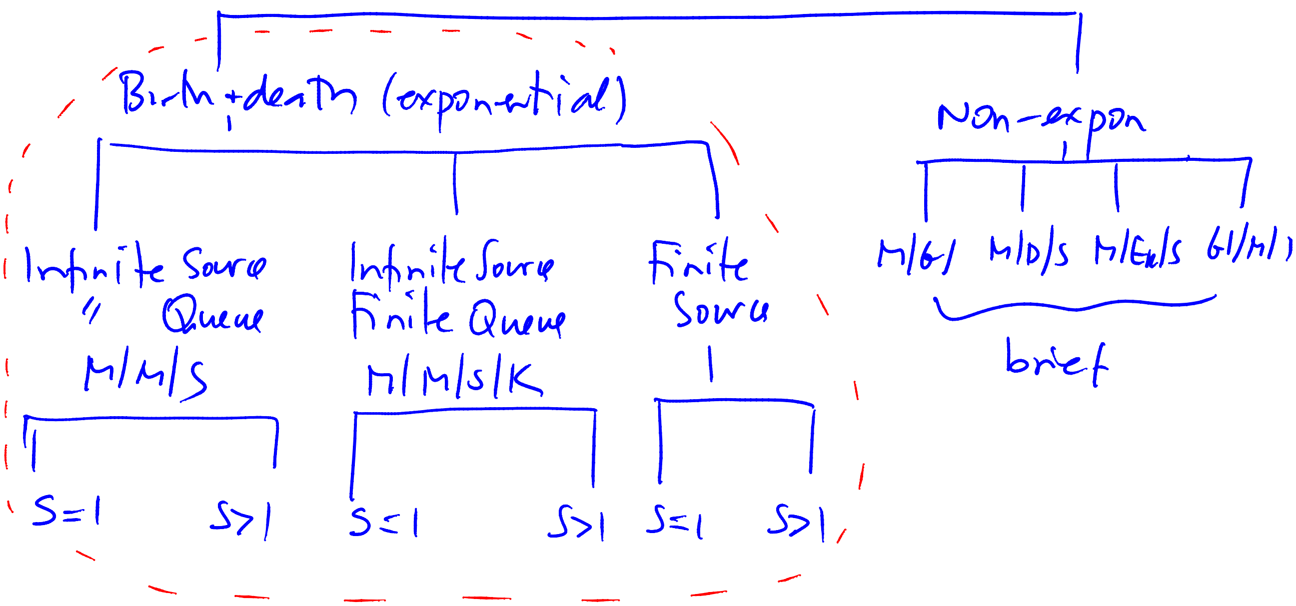


$f(t) = \lambda e^{-\lambda t}$, $\lambda = 2.47$

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i}$$

Preview

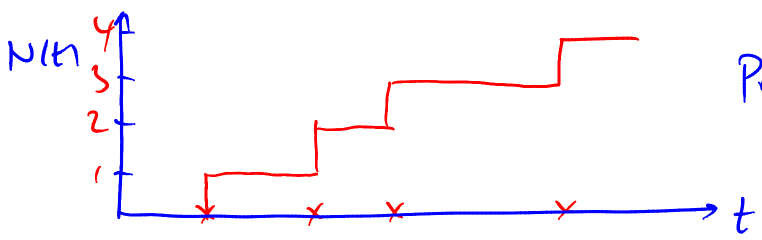
Queueing Models



Special case of Birth+death

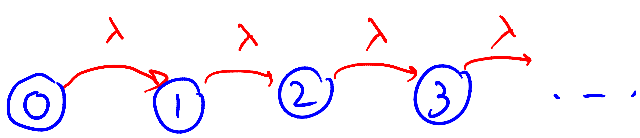
3. Birth + death process

"Pure birth"



Poisson rate λ

$$\Pr(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$



transition rate diagram

$$\Pr(\text{a "birth" in } (t, t+\Delta t)) \simeq \lambda \cdot \Delta t$$

In state n, arrival rate λ_n

" " n, departure " μ_n

In general,

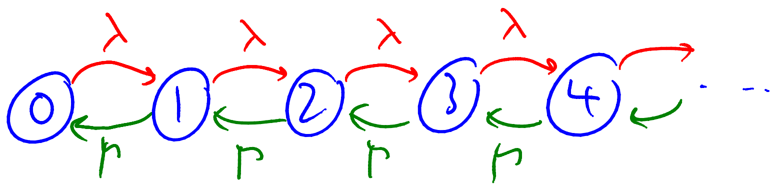
In general,

$$\Pr(\text{a birth in } (t, t+\Delta t)) \simeq \lambda_n \Delta t$$

$$\Pr(\text{a death in } (t, t+\Delta t)) \simeq \mu_n \Delta t$$

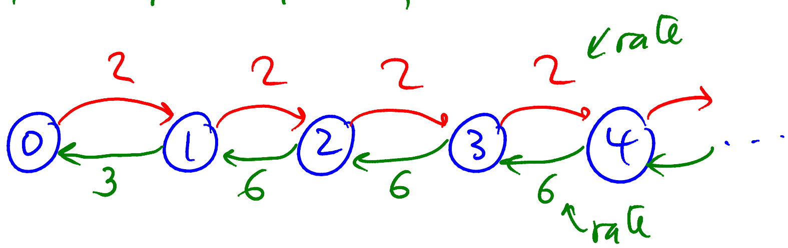
Ex,

Assume $\lambda_n = \lambda$, $\mu_n = \mu$

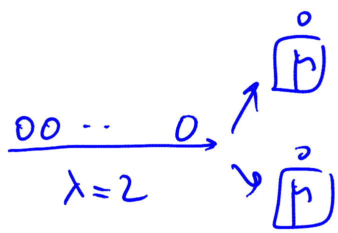


M/M/1

Ex.

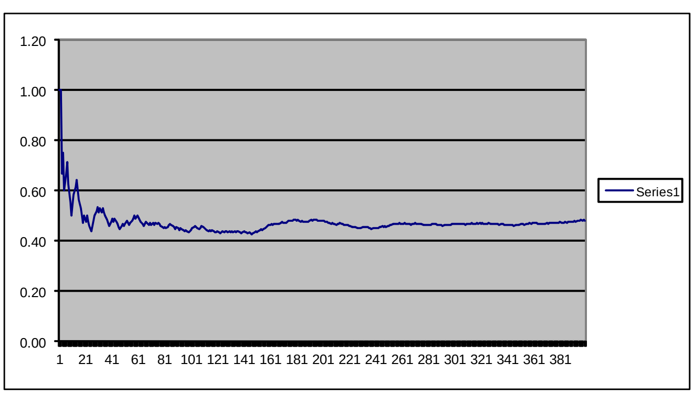


M/M/2



Each $\mu = 3$

Define, $p_n = \Pr(n \text{ cust's in system})$.



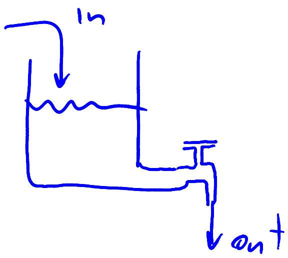
Coin toss

Rule to find p_n: Flow balance equation

State n:

Rate of departures from n = Rate of entry into n

or, Rate out = Rate in, $n = 0, 1, 2, \ldots$

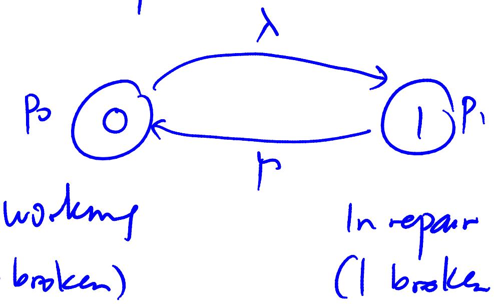


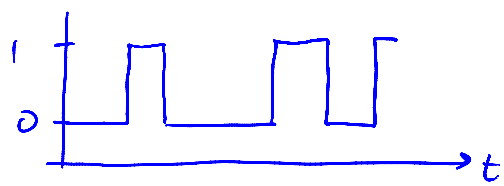
Ex. Simplest case $M/M/1/1/1$ (cap, pop. size)

Pop'n: One machine

Server: " repairman

$p_0 + p_1 = 1$





Rate Out = Rate In

$n=0$: $\boxed{\lambda p_0 = \mu p_1}$ and $\boxed{p_0 + p_1 = 1}$

$n=1$: $\mu p_1 = \lambda p_0$

Solve:

$$\left.\begin{array}{l} p_1 = \dfrac{\lambda}{\mu} p_0 \\ p_0 + p_1 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} p_0 = \dfrac{\mu}{\lambda+\mu} \\ p_1 = \dfrac{\lambda}{\lambda+\mu} \end{array}$$

$\lambda = 5,\ \mu = 10$

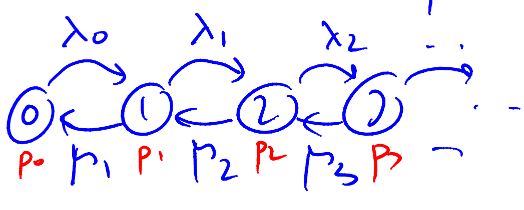
$p_0 = \dfrac{2}{3}$

$p_1 = \dfrac{1}{3}$

As λ gets large: $p_1 \to 1,\ p_0 \to 0$

As μ " " : $p_1 \to 0,\ p_0 \to 1$

Ex. General B&D process



Rate Out = Rate In

$n=0.$ $\qquad \lambda_0 p_0 = \mu_1 p_1$

$n=1$ $\qquad (\lambda_1 + \mu_1) p_1 = \lambda_0 p_0 + \mu_2 p_2$

$n=2$ $\qquad (\lambda_2 + \mu_2) p_2 = \lambda_1 p_1 + \mu_3 p_3$

$\vdots$

Sol'n

$$p_1 = \frac{\lambda_0}{\mu_1} p_0$$

$$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0$$

$$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} p_0$$

In general, $\quad p_n = C_n p_0$, where $C_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$, $n = 1, 2, \ldots$

Since $\sum^{\infty} p_n = 1 \Rightarrow$

Since $\sum_{n=0} p_n = 1 \Rightarrow$

$$p_0 = \frac{1}{1+\sum_{n=1}^{\infty} C_n} = \frac{1}{\sum_{n=0}^{\infty} C_n}, \quad C_0 \equiv 1$$

Ex. Phone operators at Queen's Park (Toronto)

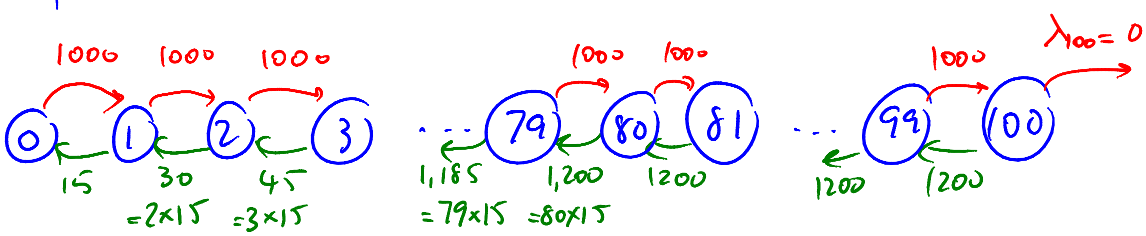
$\lambda = 1000$ calls/hr

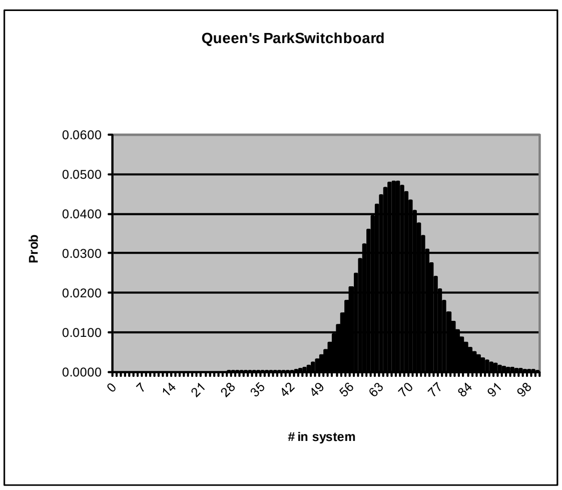
$1/\mu = 4$ min/call $\rightarrow \mu = 15$ calls/hr

80 service operators

20 calls can be on hold.

If more than $80+20=100$, lost to system





$Pr(\text{busy}) = P_{80} + \cdots + P_{100} = .07$

$Pr(\text{lost call}) = P_{100} = .00033$

$$L = \sum_{n=0}^{\infty} n P_n = 66.98, \quad L_q = \sum_{n=s}^{\infty} (n-s) P_n = 0.33$$

$$\bar{\lambda} = \sum_{n=0}^{100} \lambda_n P_n = 999.67$$

effective rate

$W = \dfrac{L}{\bar{\lambda}} = 4.02 \text{ min}$, $W_q = \dfrac{L_q}{\bar{\lambda}} = 0.018 \text{ min}$