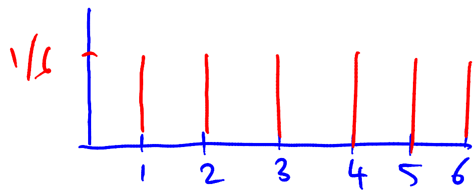
2011-09-26-EC01

Monday, September 26, 2011
6:32 PM

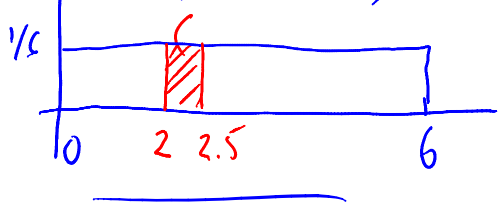
Ex. One die



$Pr(X=2) = \frac{1}{6}$

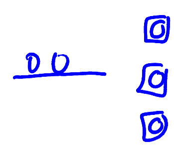
Gasoline sold

$Pr(2 \leq X \leq 2.5) =$



$Pr(X=2) = 0$

Terminology



State of system: # in system

Queue length: # in queue (# in system − # in service)

$N(t)$: # in system at $t \geq 0$

s: # servers

λ_n: arrival rate if n in system

μ_n: service " if n in system for overall system (all servers)

When $\lambda_n = \lambda$, and $\mu_n = \mu$ for all n,

$\rho = \frac{\lambda}{s\mu}$: utilization factor (traffic intensity), $0 < \rho < 1$

Two videos of a simple queueing animation:

- Stable queue: Here, the arrival rate is $\alpha = 0.45$ customers per time and service rate is $\beta = 0.50$ customers per time. Since $\alpha < \beta$, the system will eventually settle down at an average of 9 customers in the the system. The variable $Q(t)$ keeps track of the number of customers in the system, $E(t)$ measures the fraction of time the server is idle, and $W(t)$ is the average waiting time of a customer.

- Exploding queue: Here, $\alpha = 0.95$ and $\beta = 0.5$, so the queue will grow without bound.

Pasted from <http://www.business.mcmaster.ca/courses/O711/ChapterComments/Ch-17-HL.html>

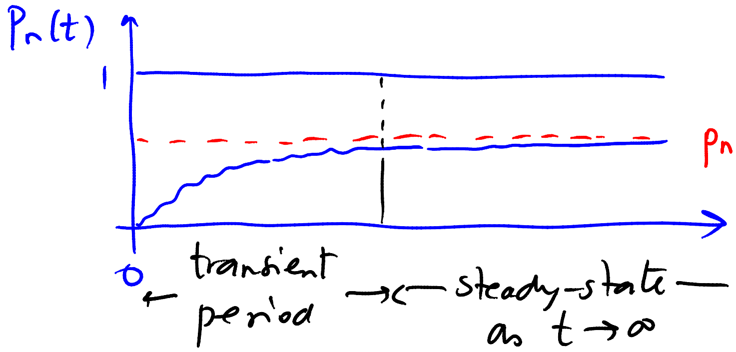
$\frac{1}{\lambda}$: E(interarrival time)

$\frac{1}{\mu}$: E(service ")

$\lambda = 20$ cust/hr

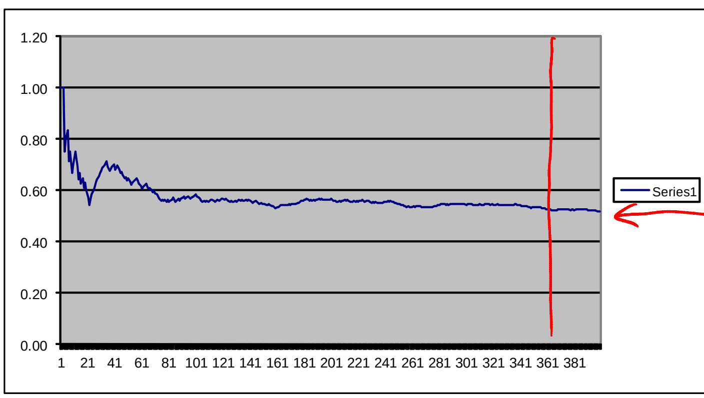
$\frac{1}{\lambda} = \frac{1}{20}$ hr/cust

$= 3$ min/cust



$p_n = \lim_{t\to\infty} P_n(t)$

$P_n(t) = Pr$(n cust in system at t)



$Pr(H) = 1/2$

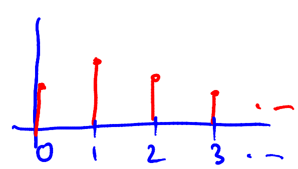
So, $p_n = Pr$(exactly n in system) in steady-state

L = E(# in system)

$X <^{a\;\; p}_{b\;\; q}$

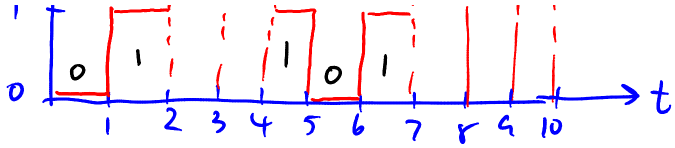
$E(X) = ap + bq$

n:	0	1	2	3	...
Pr:	p_0	p_1	p_2	p_3	...



$$L = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + \cdots = \sum_{n=0}^{\infty} n p_n$$

Ex.



0	for	2	hrs	(0.2)
1	"	3	hr	(0.3)
2	"	5	hr	(0.5)

$L = 0(0.2) + 1(0.3) + 2(0.5)$
$= 1.3$

$L_q = E(\text{queue length})$

$= 0 \cdot p_s + 1 \cdot p_{s+1} + 2 \cdot p_{s+2} + \cdots$

(with s servers)

$$= \sum_{n=s}^{\infty} (n-s) p_n$$

$s=3$

n	# in queue
3	0
4	1
5	2
$\vdots$	$\vdots$

$\mathcal{W}$: waiting time in <u>system</u> for a customer

$W = E(\mathcal{W})$

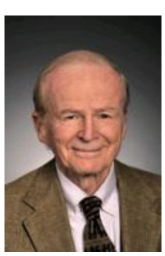
$\mathcal{W}_q$: waiting time in <u>queue</u> " " "

$W_q = E(\mathcal{W}_q)$

Easier to find L & L_q than W and W_q

<u>Little's formula</u> involves L, λ and W | $L = \lambda W$

L_q, λ and W_q | $L_q = \lambda W_q$

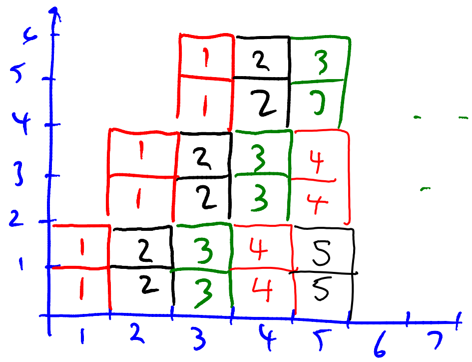


Motivating ex (Small hospital admission)

$\lambda = 2$ patients/day

$W = 3$ days

$L = \lambda W = 2 \cdot 3 = 6$ beds



2. Importance of exponential distr. in queueing

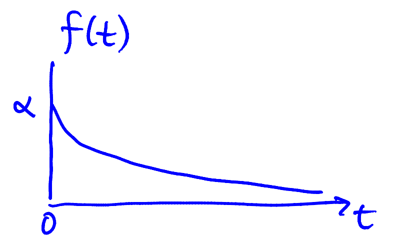
Randomness : ① interarrival time

② service time

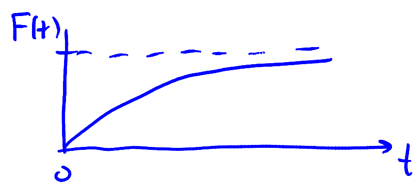
http://www.business.mcmaster.ca/courses/O711/ChapterComments/documents/ExponentialtyTest.pdf

T: exponential r.v.

Density (pdf) $f(t) = \alpha e^{-\alpha t}$, $t \geq 0$, $\underset{\text{rate}}{\alpha} > 0$



c.d.f. $\Pr(T \leq t) = F(t) = 1 - e^{-\alpha t}$



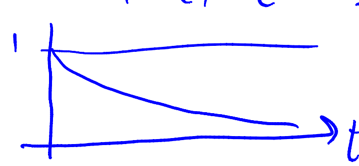
Mean $E(T) = \frac{1}{\alpha}$

$\boxed{\begin{array}{l} \alpha = 20 \text{ cust/hr} \\ \frac{1}{\alpha} = 3 \text{ min/cust} \end{array}}$

Variance $Var(T) = \frac{1}{\alpha^2}$

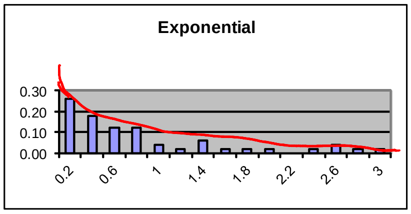
Complementary c.d.f. $\Pr(T > t) = 1 - \Pr(T \leq t)$

$= 1 - (1 - e^{-\alpha t}) = e^{-\alpha t}$



Ex. Data

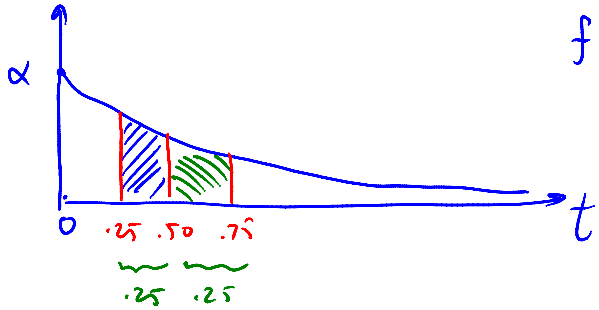
http://www.business.mcmaster.ca/courses/O711/ChapterComments/documents/Exponential.xls



Six important properties

Prop. 1 $f(t) \searrow$ in t

$f'(t) = -\alpha^2 e^{-\alpha t} < 0$

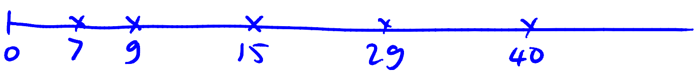


Prop. 2 Memorylessness (lack of memory)

Ex. Lifetime of an equipment

Let A: lifetimes exponential with

mean $E(A) = 10$ days/item, $\alpha = \frac{1}{10}$ items/days

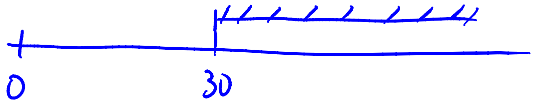


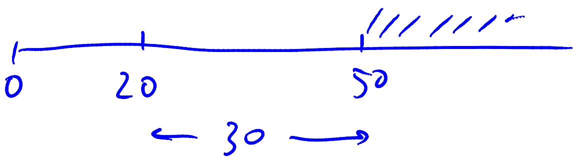
Find:

(a) Pr(no breakdown in 30 days) = Pr(A > 30) (new item)

(b) Pr(no breakdown during next 30 days | no breakdown in last 20 days)

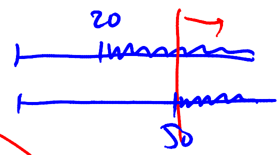
= Pr(A > 50 | A > 20)

(a) 

(b) 

$\alpha = \frac{1}{10}$

(a) $\Pr(A > \overset{t}{30}) = e^{-\frac{1}{10}\cdot 30} = e^{-3} = 0.049$



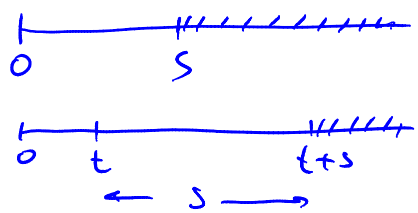
(b) $\Pr(A > 50 \mid A > 20) = \dfrac{\Pr(A > 50 \text{ and } A > 20)}{\Pr(A > 20)}$

$$= \frac{\Pr(A > 50)}{\Pr(A > 20)} = \frac{e^{-5}}{e^{-2}} = e^{-3} = 0.049$$

Always true!

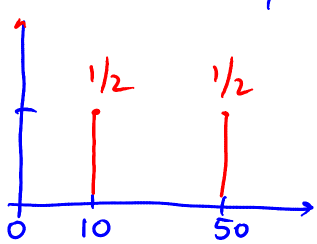
$$\Pr(A > t+s \mid A > t) = \Pr(A > s) \quad \text{for all } s, t \geq 0$$

Proof (PhD guys)



Ex. A distrib. with memory

Buses arrive either every 10 mins or 50 minutes with equal prob



$\Pr(A = 10) = \Pr(A = 50) = \frac{1}{2}$

Sample

$\Pr(\text{no arrival in 40 mins}) = \Pr(A > 40) = \frac{1}{2}$

$\Pr\left(\text{no arrival in next 40 mins} \mid \text{no arrival in 15 mins}\right) = 0$ # (Ph.D. guys)

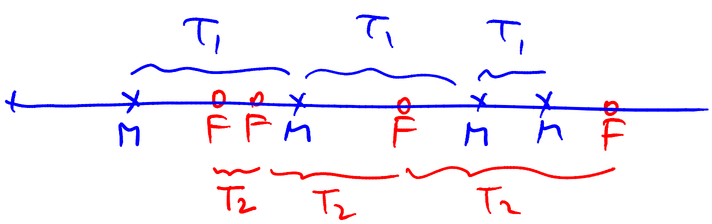
So, A has memory

Prop 3 The minimum of several independent exponentials is also exponential

Ex. Two typs: male, femals

T_1: interarrival r.v. for mals $f_1(t) = \alpha_1 e^{-\alpha_1 t}$

T_2: " " " femals $f_2(t) = \alpha_2 e^{-\alpha_2 t}$



Suppose someone just arrived (M or F). What's the distrib of time until next arrival?

$U = \min(T_1, T_2)$: time until someone arriving next (M or F)

$\alpha = \alpha_1 + \alpha_2$

$\Pr\{U > t\} = e^{-(\alpha_1+\alpha_2)t}$ (Ph.D. guys)

$\therefore$ U is also expon. with parameter $\alpha_1 + \alpha_2$

$\therefore$ U is also expon. with parameter $\alpha_1+\alpha_2$

$$\Pr\{U \le t\} = 1 - e^{-(\alpha_1+\alpha_2)t}$$

$$E(U) = \frac{1}{\alpha_1+\alpha_2}$$