c.d.f

x	p.m.f. $f(x)$	c.d.f $F(x)$
0	1/4	1/4
1	1/2	3/4
2	1/4	1

$$F(x) = \sum_{\substack{\text{all } y \\ \text{such that} \\ y \le x}} f(y)$$

$$F(0) = f(0) = \frac{1}{4}$$

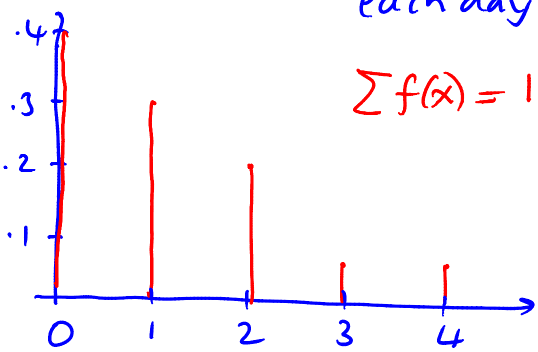
$$F(1) = f(0) + f(1) = \frac{3}{4}$$

$$F(2) = f(0) + f(1) + f(2) = 1$$

Ex. Bicycle sales @ Pierik's

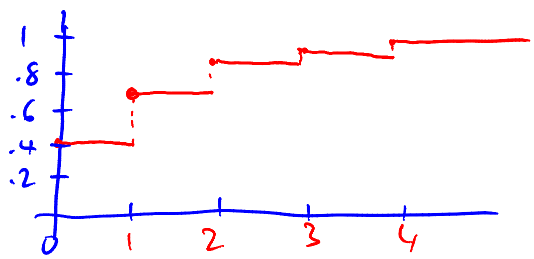
X: # bicycles sold each day

x	$Pr(X=x) = f(x)$
0	.40
1	.30
2	.20
3	.05
4	.05
	1.00



$\sum f(x) = 1$

x	$f(x)$	$F(x)$
0	.4	.4
1	.3	.7
2	.2	.9
3	.05	.95
4	.05	1.00

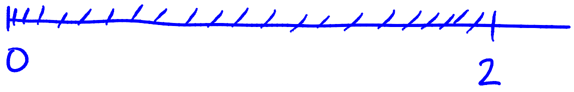


$$Pr(1 \le X \le 3) = f(1) + f(2) + f(3) = .55$$

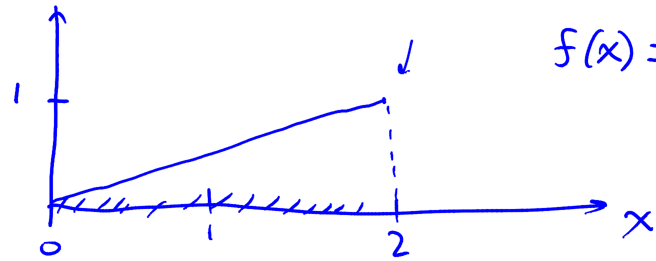
6.2. Continuous r.v.'s

Ex. Local Esso

X: # liters of gasoline sold / day (1000's)



Prob. density function



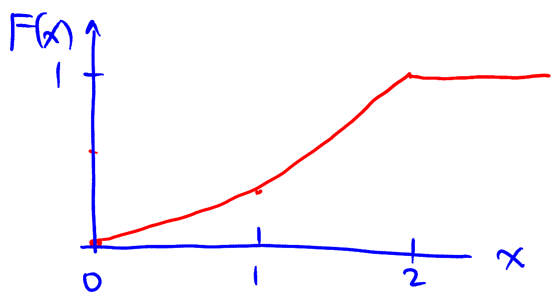
$f(x) = \frac{1}{2}x \;, \quad 0 \le x \le 2$

Make sure that "Area under $f(x) = 1$"

$$\int_0^2 f(x)\,dx = \int_0^2 \frac{1}{2}x\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right|_0^2 = 1$$

$$Pr(1 \le X \le 2) = \int_1^2 \frac{1}{2}x\,dx = \frac{3}{4}$$

c.d.f $\quad F(x) = \int_0^x f(y)\,dy = \int_0^x \frac{1}{2}y\,dy = \frac{1}{4}x^2, \quad 0 \le x \le 2$



c) Mean & variance

"central" location — dispersion around mean

Discrete

Mean : $\quad E(X) = \sum x f(x)$

Variance $\quad Var(X) = \sum [x - E(X)]^2 f(x)$

Variance.
$$Var(X) = \sum [x - E(X)]^2 f(x)$$
$$= E(X^2) - [E(X)]^2$$

Ex. Pizik

$$E(X) = 0\cdot(.40) + 1\cdot(.30) + 2\cdot(.20) + 3(.05) + 4(.05)$$
$$= 1.05 \leftarrow \text{long-run avg /day}$$

Ex.

x	$p(x)$	$xp(x)$
0	.1	
1	.1	
2	.4	
3	.3	
4	.1	

$$E(X) = 2.2$$

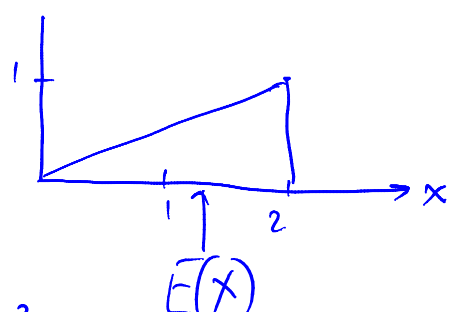
Continuous

Mean $\quad E(X) = \int_0^\infty x f(x)\,dx$

Variance $\quad Var(X) = E(X^2) - [E(X)]^2$

Ex. Esso $\qquad f(x) = \frac{1}{2}x, \quad 0 \le x \le 2$

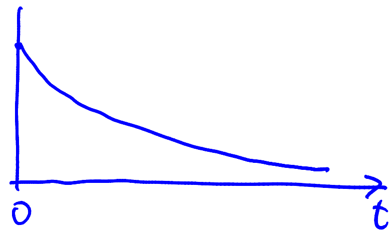
$$E(X) = \int_0^2 x \cdot \tfrac{1}{2}x\,dx = 1.33$$



$$= \int_0^2 \tfrac{1}{2}x^2\,dx = \tfrac{1}{2}\int_0^2 x^2\,dx$$
$$= \tfrac{1}{2}\left[\frac{x^3}{3}\Big|_0^2\right] = \tfrac{1}{2}\left[\frac{2^3 - 0^3}{3}\right]$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{3}(8) = \frac{8}{6} = 1.33$$

d) Important Distrib.

Exponential



- Time between arrivals
- Service time

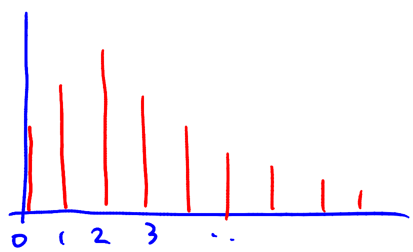
$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$

$E(\text{time}) = \dfrac{1}{\lambda}$

Formulas

$$\left\{ \begin{array}{ll} \int_a^b 1 \cdot dx = (b-a), & \int_a^b x\,dx = \frac{1}{2}(b^2 - a^2) \\ \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3), & \int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1}) \end{array} \right\}$$

Poisson



$N(t)$: # people arriving in t time unit

$$Pr[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

$n = 0, 1, 2, \ldots$

Ex. Random demand

100 units in stock

X: demand r.v

S: actual # sold

X	90	100	110
Prob	1/3	1/3	1/3
S	90	100	100

$Pr(S = 90) = 1/3$

$Pr(S = 100) = 2/3$

$E(S) = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 100 = 96.67$

$Var(S) = 22.2 \quad \leftarrow$ Show!

U: unfilled demand

X	90	100	110
U	0	0	10
Prob	1/3	1/3	1/3

$\underbrace{1/3 \quad 1/3}_{2/3}$

So, $Pr(U=0) = \frac{2}{3}$

$Pr(U=10) = \frac{1}{3}$

$E(U) = 0 \cdot \frac{2}{3} + 10 \cdot \frac{1}{3} = \frac{10}{3} = 3.33$

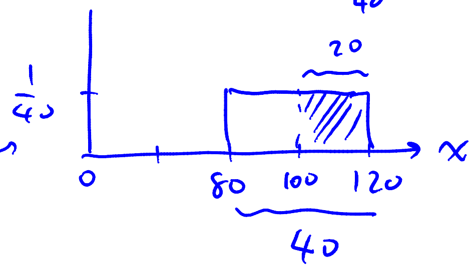
$Var(U) = 22.2$ ← show

Ex. Same problem except cont. demand

100 in stock

$f(x) = \frac{1}{40}, \quad 80 \le x \le 120$

$40 \cdot \frac{1}{40} = 1$



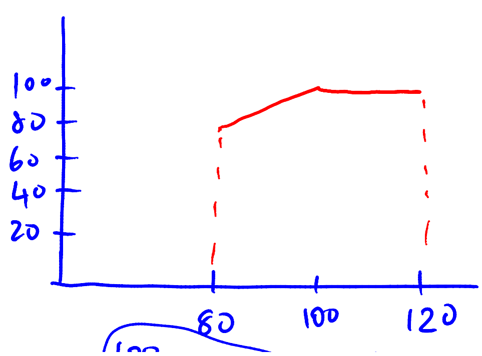
$Pr(X > 100) = \frac{1}{2}$

↗ shortage

S: sales

$$S = \begin{cases} 100, & X > 100 \\ x & X \le 100 \end{cases}$$

Sales



$$E(S) = \int_{80}^{100} x \frac{1}{40}\,dx + \int_{100}^{120} 100 \cdot \frac{1}{40}\,dx = 95$$

$Var(S) = 41.67$ ← show

$$\frac{1}{40}\int_{80}^{100} x\,dx = \frac{1}{40}\,\frac{x^2}{2}\Big|_{80}^{100}$$

$$= \frac{1}{80}\left(100^2 - 80^2\right)$$

Ex. Lady Gaga on my iPod

100 songs → shuffle

L: Lady Gaga on first song

$Pr(L) = \frac{1}{100}$, $Pr(\bar{L}) = Pr(\text{not } L) = \frac{99}{100}$

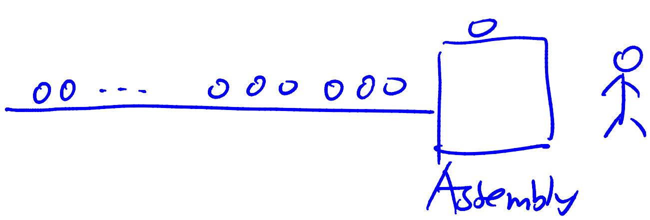
Three shuffles

$$Pr(\text{at least one } L) = 1 - Pr(\text{no } L)$$

$$= 1 - \left(\frac{99}{100}\right)^3$$

Ch.17 Queueing Applications

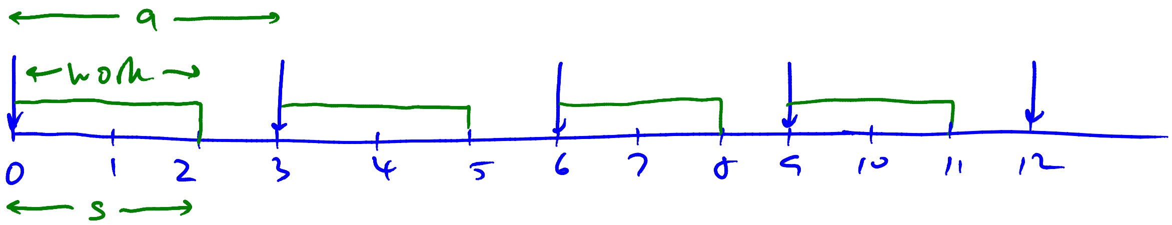
Ex. Naive deterministic model (assembly line)



Let $a = 3$ min/item (time between arrivals): $\overset{\lambda}{\text{rate}} = 20/\text{hr}$

$s = 2$ min/item (service time) ; rate $= 30/\text{hr}$

r



% time idle $= \frac{1}{3}$: $\frac{a-s}{a}$

% time busy $= \frac{2}{3}$: $\frac{s}{a}$

If $a > s$: Idle $1 - s/a$ of time

$a = s$. fully busy

$a < s$. Queue explodes

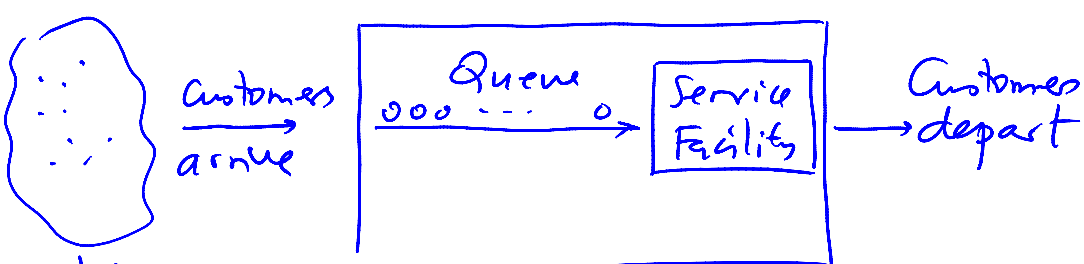
Ex. I Love Lucy

http://www.youtube.com/watch?v=uztA6JCKB4s

Ex. Arrival + service process with variations

http://www.business.mcmaster.ca/courses/O711/ChapterComments/documents/VariationsInArrivalService.pdf

1. Features of queueing process

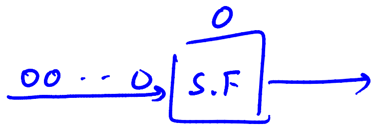


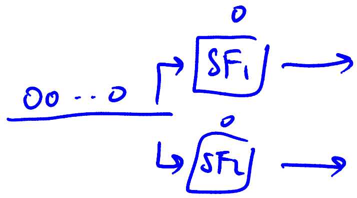
Input source
- finite or infinite
- Single or bulk arrivals
- Control of arrivals
- Arrival distrib.

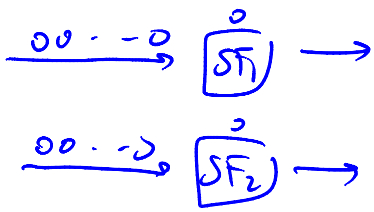
Queue discipline
- FCFS
- LCFS
- Priority
- Balking

Output process
- # servers
- Parallel "
- service distrib

· Reneging
· Jockeying

Single server  One barber shop
Gas station (1 pump)

Multi-server  Banks

Multi-server  Grocery stores

1950s 1970s

<u>Notation</u>: (Kendall-Lee notation)

___ / ___ / ___

Interarrival times — service times — # servers

M : exponential (memoryless)
D : "degenerate" (constant)
E_k : Erlang
G : general (GI: general input)

Ex. M/M/1, M/D/1, GI/D/3

Note: If system capacity is finite, use a 4th symbol

$-/-/-/K$

↑ system cap.

Barber

$M/M/3/8$

□ □ □

Tony Bruno Joseph

○ ○ ○ ○ ○