2013-12-09

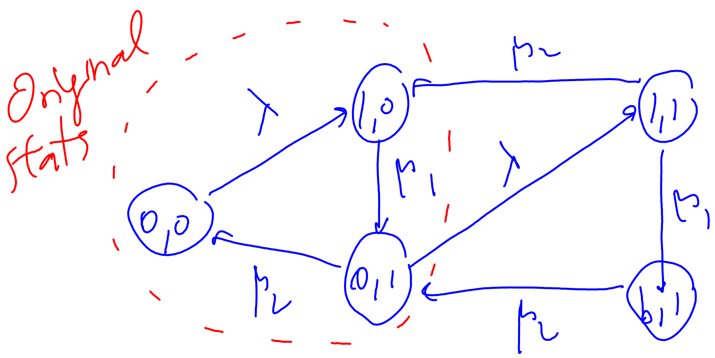
Monday, December 09, 2013
10:31 AM

Ex. Shoeshine establ. w/ extra capacity

clean shine

(x, y)

	States		
	Chair 1	Chair 2	Description
$(0,0)$	0	0	System empty
$(1,0)$	1	0	1 busy, 2 empty
$(0,1)$	0	1	1 empty, 2 busy
$(1,1)$	1	1	1 busy, 2 busy
$(b,1)$	b	1	1 "blocked", 2 busy but idle



$$
Q = \begin{array}{c|ccccc}
 & 00 & 10 & 01 & 11 & b1 \\
\hline
00 & -\lambda & \lambda & & & \\
10 & & -\mu_1 & \mu_1 & & \\
01 & \mu_2 & & -(\lambda+\mu_2) & \lambda & \\
11 & & \mu_2 & & -(\mu_1+\mu_2) & \mu_1 \\
b1 & & & \mu_2 & & -\mu_2
\end{array}
$$

Balance eqn's

Rate Out = Rate In

balance eqns

Rate Out = Rate In

States	
(0,0)	$\lambda p_{00} = \mu_2 p_{01}$
(0,1)	$(\lambda+\mu_2) p_{01} = \mu_1 p_{10} + \mu_2 p_{b1}$
(1,0)	$\mu_1 p_{10} = \lambda p_{00} + \mu_2 p_{11}$
(1,1)	$(\mu_1+\mu_2) p_{11} = \lambda p_{01}$
(b,1)	$\mu_2 p_{b1} = \mu_1 p_{11}$

$$p_{00}+p_{01}+p_{10}+p_{11}+p_{b1}=1$$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Shoeshine-Extra.mw

$\lambda = 10, \quad \mu_1 = 20, \quad \mu_2 = 30$

$$p_{00}, \ldots \quad \begin{cases} 0.4891304348 \\ 0.1630434783 \\ 0.2934782609 \\ 0.03260869565 \\ 0.02173913043 \end{cases}$$

$\overline{}$ 1.0

$$\lambda_{eff} = \lambda(p_{00}+p_{01}) = 6.6 \quad (\text{vs. } 5.5)$$

$$L = 1\cdot(p_{01}+p_{10}) + 2\cdot(p_{11}+p_{b1}) = .56 \quad (\text{vs } .45)$$

$$W = \frac{L}{\lambda_{eff}}$$

v v

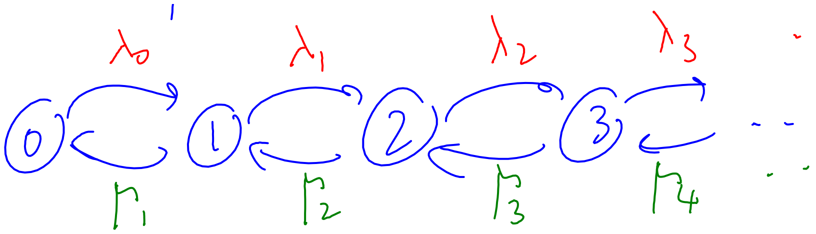
A a

D d

X x

$E(D) = \int d f(d)\, dd$

Ex. B&D process



State	Rate Out	=	Rate In
0	$\lambda_0 P_0$	$=$	$\mu_1 P_1$
$n \geq 1$	$(\lambda_n + \mu_n) P_n$	$=$	$\mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$

Sol'n:

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$$

$$\vdots$$

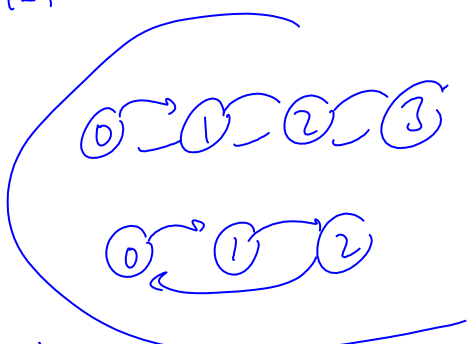
$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\prod_{i=0}^{n-1} \lambda_i \cdot P_0}{\prod_{i=1}^{n} \mu_i}, \quad (n \geq 1)$$

$$\vdots$$

Use $\sum_{n=0}^{\infty} P_n = 1$

$$\Downarrow$$

$$P_0 + P_1 + P_2 + \cdots = 1$$



$$\Rightarrow \quad P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right]^{-1}$$

must be $< \infty$

$\prod \lambda_i$

$\left(\sum_{n=1} \prod_{i=1} \mu_i \right)$ must be $< \infty$

$$\Rightarrow \quad P_n = \frac{\prod \lambda_i}{\prod \mu_i \left(1 + \sum_{1}^{\infty} \frac{\prod \lambda_i}{\prod \mu_i} \right)}, \quad n \geq 1$$

Specializing: $\left. \begin{array}{l} \lambda_n = \lambda \\ \mu_n = \mu \end{array} \right\}$ M/M/1

$$P_n = \left(\frac{\lambda}{\mu} \right)^n \left(1 - \frac{\lambda}{\mu} \right), \quad n \geq 0$$

$$\rho = \frac{\lambda}{\mu}, \quad 0 < \rho < 1$$

Ex. Transient Sol'n to M/M/1 queue $(X(0) = 0)$

$$P_n(t) = e^{-(\lambda+\mu)t} \left\{ \rho^{n/2} I_n(at) + \rho^{(n-1)/2} I_{n+1}(at) + (1-\rho)\rho^n \sum_{k=n+2}^{\infty} \rho^{-k/2} I_k(at) \right\}$$

$\rho = \frac{\lambda}{\mu}, \quad a = 2\sqrt{\lambda \mu}$ (Saaty, p. 340)

$I_n(at)$: BesselI(n, at): Modified Bessel function of 1st kind http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Transient-MM1.mw

$f := x^2$ $\qquad \cancel{f(x)}$ $\qquad f(x) := x^2$

$f := x \to x^2$ $\qquad f(2) = 4$

$f := \text{unapply}(x^2, x)$

Ex. Pure birth (Poisson)

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[r, bend left, "\lambda"] & \cdots \end{tikzcd}$$

Forward DE

$$P'_{ij}(t) = -\lambda P_{ij}(t) + \lambda P_{i,j-1}(t)$$

as $t \to \infty$ (?) $\qquad 0 = -\lambda P_j + \lambda P_{j-1}$ $\qquad$ (Correct?)

$\Rightarrow \quad P_j = P_{j-1}$ $\qquad$ limit doesn't exist

$\sum P_j = 1$

$\lambda = 1$

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[r, bend left, "1"] & 2 \arrow[r, bend left, "1"] & \cdots \end{tikzcd}$$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \cdots \\ \hline 0 & & 1 & & \\ 1 & & & 1 & \\ 2 & & & & 1 \end{array}$$

Not irreducible

Is recurrent

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}$$

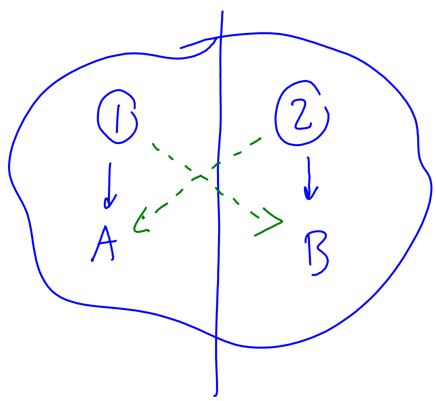
Not irreducible

is recurrent

Ex. Response areas for two ambulances

Two units (e.g., ambulances) 1, 2

Two regions $j = A, B$



1 for A (or B)
2 for B (or A)

Arrivals from j (A, B) is Poisson with λ_j

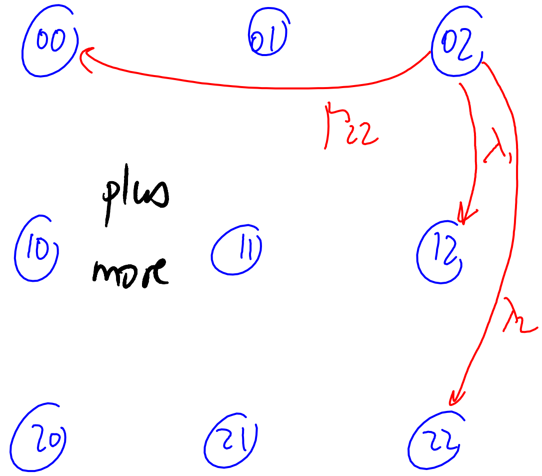
Service exponential with μ_{ij} $(i = 1, 2,\ j = A, B)$

$X_i(t)$: status of unit i at t → $=0$ idle
$=1$ at A
$=2$ at B

$X(t) = (X_1(t), X_2(t))$

$\setminus X_{2,1}$

$X_1 \backslash X_2$	0	1	2
0	00	01	02
1	10	11	12
2	20	21	22



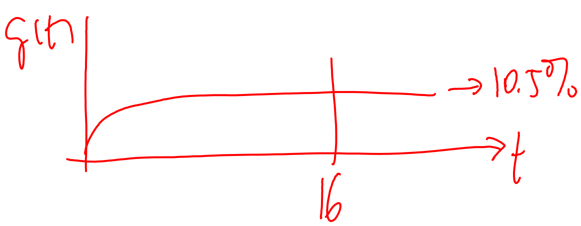
let $\lambda_1 = .1 \quad \mu_{11} = .5 \quad \mu_{12} = .25$ rates/hr

$\lambda_2 = .125 \quad \mu_{21} = .2 \quad \mu_{22} = .4$

$$p_{ij}(t) = Pr\{X_1(t) = i,\ X_2(t) = j\}$$

$$q(t) = \sum_{i=1}^{2} \sum_{j=1}^{2} p_{ij}(t) \quad : Pr(\text{call lost})$$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Ambulances.mw



Transient solution to the CTMC

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http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/documents/Transient-CTMC.PDF