2013-12-02

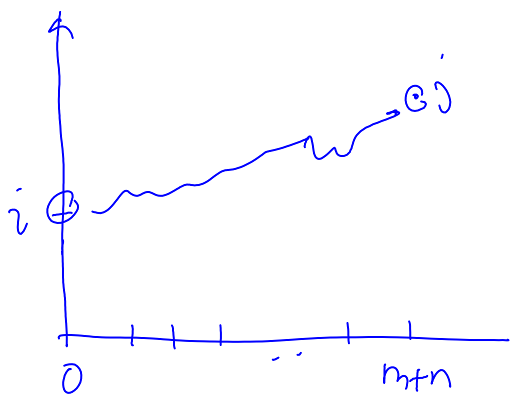
Monday, December 02, 2013
1:38 PM

Dec. 2 today
" 9 make-up
Dec. 12 Hw2 due $\to$ Dec. 13 Project due
Dec. 16 final

a) Kolmogorov D.E.

In DTMC $\quad P^{(n)} = P^{(n-1)} \cdot P = P^n$

$$p_{ij}^{(n)} = Pr\{X_n = j \mid X_0 = i\}$$



Fact: $\quad p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} p_{kj}^{(n)}$ (show, easy!)

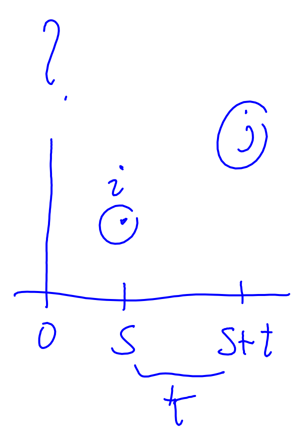
$$\Rightarrow \quad P^{(m+n)} = P^{(m)} P^{(n)}$$

In CTMC?

$$p_{ij}(t+s) = Pr\{X(t+s) = j \mid X(0) = i\} \quad ?$$

Define $\quad p_{ij}(t) = Pr\{X(t+s) = j \mid X(s) = i\}$

$$= Pr\{X(t) = j \mid X(0) = i\}$$



Three lemmas needed

<u>Lemma 1</u> $$\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i$$

<u>Proof</u> $P_{ii}(h) = \Pr\{X(h)=i \mid X(0)=i\}$

$1-P_{ii}(h) = \Pr\{X(h) \text{ anywhere but } i \mid X(0)=i\} =$

$= \Pr\{\text{a transition in } h \mid X(0)=i\} = v_i h + o(h)$

$$\therefore \lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = \lim \frac{v_i h + o(h)}{h} = v_i$$

v_i

q_{ij} P_{ij}

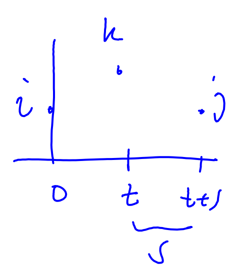
<u>Lemma 2</u> $$\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij} \quad (= v_i P_{ij})$$

<u>Proof</u> $P_{ij}(h) = \Pr\{X(h)=j \mid X(0)=i\}$

$= \underbrace{v_i}_{\text{leave } i} \underbrace{P_{ij}}_{\text{to } j \text{ from } i} h + o(h)$

$$\therefore \lim_{h\to 0} \frac{P_{ij}(h)}{h} = \lim \frac{v_i P_{ij} h + o(h)}{h} = v_i P_{ij} = q_{ij}$$

<u>Lemma 3</u> Chapman-Kolmogorov eqn's



For $s \geq 0$, $t \geq 0$

prove

$$\Pr\{X(t+s)=j \mid X(0)=i\} = P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

Ex Three states 1, 2, 3

Ex. Three states 1, 2, 3

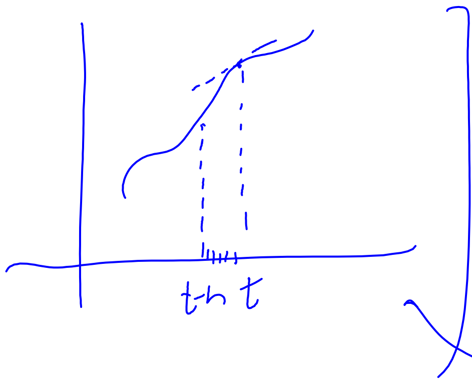
$i=1, j=3$

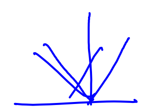
$$P_{13}(t+s) = \sum_{k=1}^{3} P_{1k}(t)\,P_{k3}(s) = P_{11}(t)\,P_{13}(s) + P_{12}(t)\,P_{23}(s) + P_{13}(t)\,P_{33}(s)$$

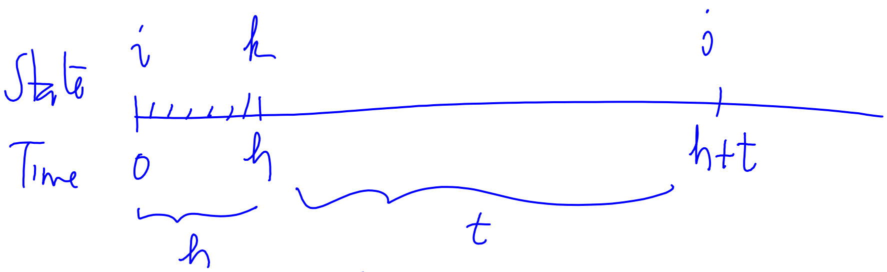
	1	2	3
1	·	·	✓
2	·	·	·
3	·	·	·

i) Kolmogorov Backward Equations

In calculus







$$\lim_{h\to 0} \frac{f(t)-f(t-h)}{h}$$

$$p_{ij}(h+t) = \sum_{k=0}^{\infty} p_{ik}(h)\,p_{kj}(t)$$

$$p_{ij}(h+t) - p_{ij}(t) = \sum_{k=0}^{\infty} p_{ik}(h)\,p_{kj}(t) - p_{ij}(t)$$

$$= \sum_{k=0}^{i-1} p_{ik}(h)\,p_{kj}(t) + p_{ii}(h)\,p_{ij}(t) + \sum_{k=i+1}^{\infty} p_{ik}(h)\,p_{kj}(t) - p_{ij}(t)$$

$$= \sum^{\infty} p_{ik}(h)\,p_{kj}(t) - [1 - p_{ii}(h)]\,p_{ij}(t)$$

$$= \sum_{k=0}^{i-1} p_{ik}(h)p_{kj}(t) + p_{ii}(h)p_{ij}(t) + \sum_{k=i+1}^{\infty} p_{ik}(h)p_{kj}(t) - p_{ij}(t)$$

$$= \sum_{\substack{k=0 \\ k\neq i}}^{\infty} p_{ik}(h)p_{kj}(t) - [1-p_{ii}(h)]\,p_{ij}(t)$$

Divide by h and let $\to 0$ Meczla

$$\frac{p_{ij}(t+h)-p_{ij}(t)}{h} = \frac{\sum_{\substack{k=0 \\ k\neq i}}^{\infty} p_{ik}(h)\,p_{kj}(t)}{h} - \frac{[1-p_{ii}(h)]}{h}\,p_{ij}(t)$$

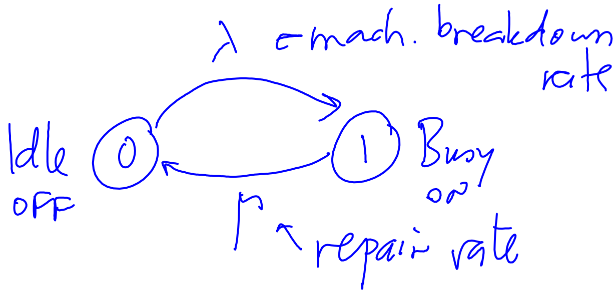
let $h\to 0$

$$p'_{ij}(t) = \sum_{k\neq i}^{\infty} q_{ik}\,p_{kj}(t) - v_i\,p_{ij}(t), \qquad p_{ij}(0)=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

(Kronecker's delta)

Ex. Two state CTMC

Repairman



M/M/1/1

$P_{00}(8) = ?$

$v_0 = \lambda, \quad q_{01} = \lambda$

$v_1 = \mu, \quad q_{10} = \mu$

$P_{00}(0) = 1$

$q_{01} = v_0 P_{01}$ ⟵ 1

$q_{10} = v_1 P_{10}$ ⟵ 1

$$P'_{00}(t) = q_{01}P_{10}(t) - v_0 P_{00}(t)$$

① $\boxed{P'_{00}(t) = \lambda P_{10}(t) - \lambda P_{00}(t) \qquad P_{00}(0)=1}$

① or $\boxed{P_{00}'(t) = \lambda P_{10}(t) - \lambda P_{00}(t) \quad , \quad P_{00}(0) = 1}$

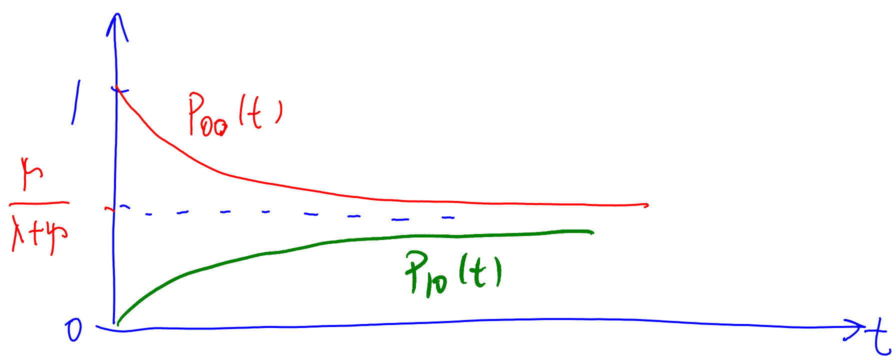
② $P_{10}'(t) = q_{10} P_{00}(t) - v_1 P_{10}(t)$

or $\boxed{P_{10}'(t) = \mu P_{00}(t) - \mu P_{10}(t) \quad , \quad P_{10}(0) = 0}$

Solving this system,

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad , \qquad P_{01}(t) = 1 - P_{00}(t)$$

$$P_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad , \qquad P_{11}(t) = 1 - P_{10}(t)$$



Matrix form

(Show)

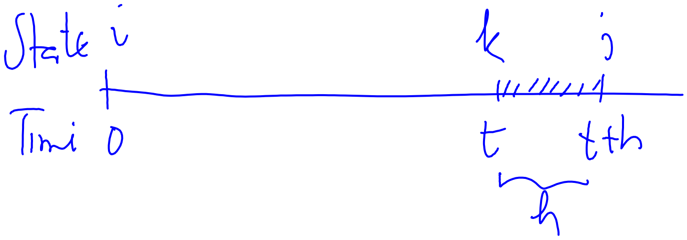
$$\begin{bmatrix} \boxed{P_{00}'(t)} & P_{01}'(t) \\ P_{10}'(t) & P_{11}'(t) \end{bmatrix} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \begin{bmatrix} P_{00}(t) & P_{01}(t) \\ P_{10}(t) & P_{11}(t) \end{bmatrix}$$

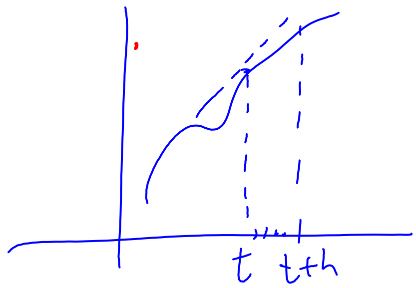
$$P'(t) = (\;)\,P(t) \quad , \quad P(0) = I$$

(Backward

(Q behind P

ii) Kolmogorov's forward eqn's





$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = f'(t)$$

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

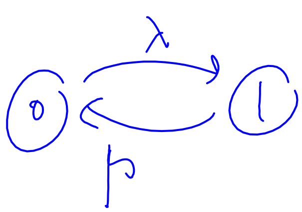
Pull out $k=j$

$$= \sum_{k\neq j} P_{ik}(t) P_{kj}(h) \underbrace{- P_{ij}(t) + P_{ij}(t) P_{jj}(h)}_{-[1-P_{jj}(h)] \quad P_{ij}(t)}$$

Divide by h and let $h \to 0$

$$\boxed{P_{ij}'(t) = \sum_{k\neq j} P_{ik}(t) q_{kj} - v_j P_{ij}(t), \quad P_{ij}(0) = \delta_{ij}}$$

Ex. Two State CTMC



$v_0 = \lambda \quad q_{01} = \lambda$

$v_1 = \mu \quad q_{10} = \mu$

$$P_{00}'(t) = P_{01}(t)\mu - \lambda P_{00}(t), \quad P_{00}(0) = 1$$

$$P_{01}'(t) = P_{00}(t)\lambda - \mu P_{01}(t), \quad P_{01}(0) = 0$$

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\ )t}$$

as before!

Need a separate system

$$P_{10}(t) = \frac{\mu}{\lambda+\mu} - (\to 0)$$

$$P_{11}(t) = \frac{\lambda}{\lambda+\mu} + (\to 0)$$

Matrix form

$$\begin{bmatrix} P_{00}'(t) & P_{01}'(t) \\ P_{10}'(t) & P_{11}'(t) \end{bmatrix} = \begin{bmatrix} P_{00}(t) & P_{01}(t) \\ P_{10}(t) & P_{11}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}, \quad P(0) = I$$

$$P'(t) = P(t)Q$$

Unconditional state probs $\Pi_j(t)$

Analogous to $\Pi_j^{(n)}$ in DTMC, define

$$\Pi_j(t) = \Pr\{X(t) = j\}$$

Again using Kolm. DE

$k \to j$

$0 \quad t \quad t+h$

$$\Pi_j(t+h) - \Pi_j(t) = \sum_{k=0}^{\infty} \Pi_k(t) P_{kj}(h) - \Pi_j(t)$$

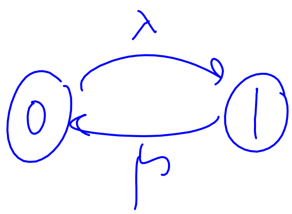
$$= \sum \Pi_k(t) P_{kj}(h) - \Pi_j(t) + \Pi_j(t) P_{jj}(h).$$

$$= \sum_{k \neq j} \Pi_k(t) P_{kj}(h) \underbrace{- \Pi_j(t) + \Pi_j(t) P_{jj}(h)}_{-[1 - P_{jj}(h)] \Pi_j(t)}$$

Divide by h and $h \to 0$

$$\Pi_j'(t) = \sum_{k \neq j} \Pi_k(t) q_{kj} - \nu_j \Pi_j(t)$$

Ex. Two state



$\nu_0 = \lambda \qquad q_{01} = \lambda$

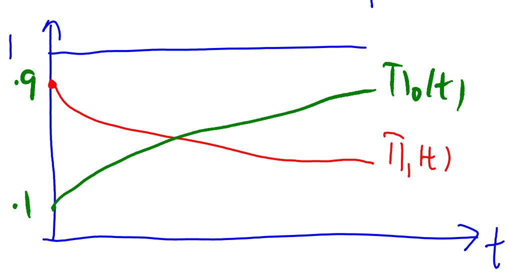
$\nu_1 = \mu \qquad q_{10} = \mu$

$$\Pi_0'(t) = \Pi_1(t)\mu - \lambda \Pi_0(t), \quad \Pi_0(0) = 0.1 \text{ (say)}$$

$$\Pi_1'(t) = \Pi_0(t)\lambda - \mu \Pi_1(t), \quad \Pi_1(0) = 0.9$$

Sol'n: $$\Pi_0(t) = \frac{\mu}{\lambda+\mu} + 0.1 \frac{\lambda}{\lambda+\mu} e^{-(\;)t} - 0.9 \frac{\mu}{\lambda+\mu} e^{-(\;)t}$$

$$\Pi_1(t) = \frac{\lambda}{\lambda+\mu} - 0.1 \frac{\lambda}{\lambda+\mu} e^{-(\;)t} + 0.9 \frac{\mu}{\lambda+\mu} e^{-(\;)t}$$



Remark. Standard notation for $\Pi_j(t)$ is $P_j(t)$

3h

Note. Recall that

$$P^{(n)} = P^{(n-1)} \cdot P$$

$$\Delta P^{(n)} = P^{(n)} - P^{(n-1)} = P^{(n-1)} \cdot P - P^{(n-1)} = P^{(n-1)} \underbrace{(P - I)}_{Q}$$

$$= P^{(n-1)} Q$$

If $P = \begin{bmatrix} .4 & .6 \\ .7 & .3 \end{bmatrix}$, $Q = P - I = \begin{bmatrix} -.6 & .6 \\ .7 & -.7 \end{bmatrix}$ rows sum to 0

Now, if we <u>define</u> $q_{ii} = -v_i$

KDE (Back)

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

$$= \sum_{k \neq i} q_{ik} P_{kj}(t) + q_{ii} P_{ij}(t) = \sum_{k=0}^{\infty} q_{ik} P_{kj}(t)$$

In matrix form

$\mathbb{P}'(t) = Q\,\mathbb{P}(t)$, $\mathbb{P}(0) = I$ Backward

$\mathbb{P}'(t) = \mathbb{P}(t)\,Q$, $\mathbb{P}(0) = I$ Forward

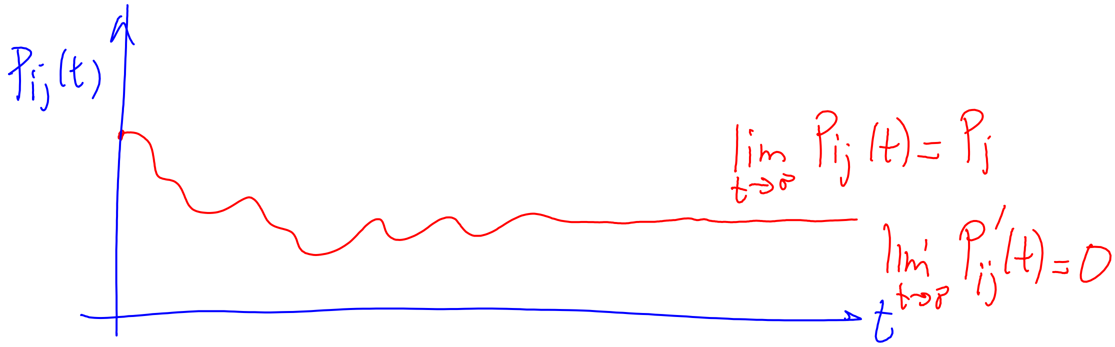
b) limiting prob's

In DTMC, $\pi = \pi P$, $\pi e = 1$

CTMC
Consider, e.g., the For-eqn's

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - v_j P_{ij}(t), \quad P_{ij}(0) = \delta_{ij}$$

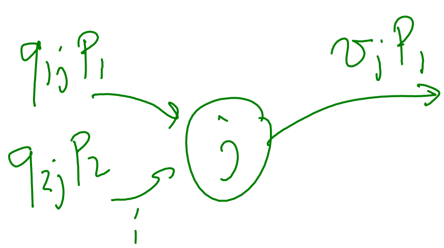
If $P_{ij}(t)$ were known, it would look so

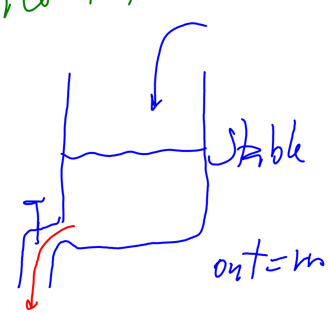


$$0 = \sum_{k \neq j} P_k q_{kj} - v_j P_j, \quad \sum P_j = 1$$

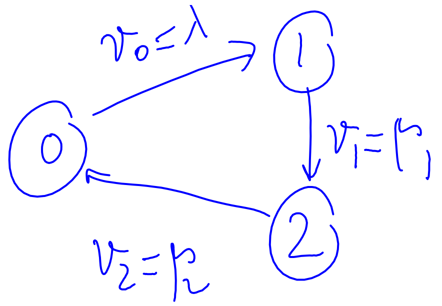
$$0 = \text{rate in} - \text{rate out}$$

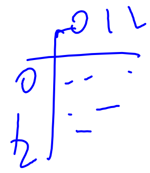
Balance eqn's: Rate Out = Rate In





General analysis of shoeshine booth





$N\times N = 3\times 3 = 9$
DEs

State	Rate Out	=	Rate In
0	λP_0	=	$\mu_2 P_2$
1	$\mu_1 P_1$	=	λP_0
2	$\mu_2 P_2$	=	$\mu_1 P_1$

$$P_0+P_1+P_2=1$$

$$P_2=\frac{\lambda}{\mu_2}P_0$$

$$P_1=\frac{\lambda}{\mu_1}P_0$$

$$P_0=\left(1+\frac{\lambda}{\mu_1}+\frac{\lambda}{\mu_2}\right)^{-1}$$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Shoeshine.mw

$\lambda=10,\ \mu_1=20,\ \mu_2=30,\qquad P_0=.55,\ P_1=.27,\ P_2=.18$

Effective arrival rate: $\lambda_{eff}=\lambda P_0=5.5$ cust/hr

Avg. # in system $L=1\cdot p_1+1\cdot p_2=.45$ cust

Avg. time in system. $W=\frac{1}{\mu_1}+\frac{1}{\mu_2}=.083$ hr

Little's formula $L=\lambda_{eff}W$ ✓

Prop. of time. Chair 1 idle: $P_0+P_2=.73$

" " " 2 " $P_0+P_1=.82$

In general

A_s $\lambda \uparrow$ $\quad P_0 \downarrow$

" $\lambda \downarrow$ $\quad Pr(1 \text{ or } 2 \text{ busy}) = P_1 \downarrow$

$P_2 \downarrow$