2013-11-25

Monday, November 25, 2013
10:44 AM

Clarification

Theorem. Consider an irreducible MC! This MC belongs to one of three classes below

possibly ∞ # states

Lucille Ball video

	Periodicity: Aperiodic	Periodic (d)
Transient / Recurrent-null	$P_{ij}^{(n)} \to 0$ and π_j don't exist (applies when MC is infinite)	
Recurrent-pos.	$P_{ij}^{(n)} \to \pi_j > 0$ unique from $\pi = \pi P$, $\pi e = 1$	$P_{jj}^{(nd)} \to d\pi_j$. Here π_j are long-run fraction of time in state j (Maple example online)

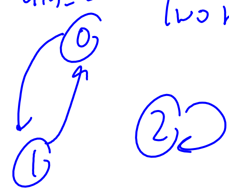
http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Stationary-General-Periods.mw

Final Dec. 16 (M) 9-12
DSB - 505

Ex.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$d(0)=2$



Two recurrent seb

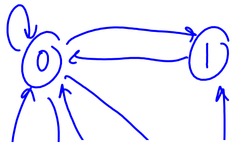
Not irreducible

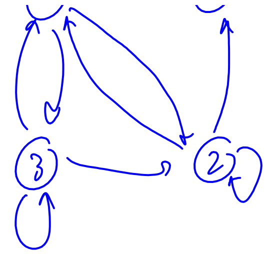
$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P^4 = P^2$, $P^5 = P^3$, etc

$\pi = \pi P$, $\pi e = 1$ no sol'n

Ex. (s,S) Inv. system





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http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Periodic-sS-Limit.mw

Classification of the states of a Markov chain (summary)

Pasted from <http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/ch-04.html>

e) MC with countably infinite states

Ex. Air conditioner repair

Repairguy accepts at most 2 jobs while working on one

# arrivals	0	1	≥ 2
Prob	p	q	r

$, \; p+q+r=1$

X_n is a MC

$\rho = \frac{\lambda}{\mu} < 1$ for stability

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \cdots \end{array} \\ \begin{bmatrix} p+q & r & 0 & 0 & \\ p & q & r & 0 & \cdots \\ 0 & p & q & r & \\ 0 & 0 & p & q & r \\ & & \ddots & & \ddots \end{bmatrix} \end{array}$$

$\Pi = \Pi P, \quad \Pi e = 1, \quad \Pi = (\Pi_0, \Pi_1, \dots)$

$$\Pi_k = \sum_{i=0}^{\infty} \pi_i P_{ik}$$

$(p+q)\Pi_0 + p\Pi_1 = \Pi_0$ $\quad 7^{\circ}$

$c'x \quad c = (\,) \quad c'x$

$$\pi_k = \sum_{i=0} \pi_i P_{ik}$$

$$(p+q)\pi_0 + p\pi_1 = \pi_0 \qquad z^0$$
$$r\pi_0 + q\pi_1 + p\pi_2 = \pi_1 \qquad z^1$$
$$r\pi_1 + q\pi_2 + p\pi_3 = \pi_2 \qquad z^2$$
$$r\pi_2 + q\pi_3 + p\pi_4 = \pi_3 \qquad z^3$$
$$\vdots$$

$c'x \qquad c=[\ ] \quad c'x$

$[1 \ \cdot\ ] = [0\ 0] \begin{bmatrix} 0 & \cdot \\ 0 & \cdot \end{bmatrix} \begin{matrix} =1 \\ =1 \end{matrix}$

100×1

$\pi = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}$ ✓

$\pi' = \pi' P$

$(\pi_0, \pi_1) = (\pi_0, \pi_1)\begin{pmatrix} & \end{pmatrix}$

$$\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i$$

$\Rightarrow$

$$p\pi_0 + rz\,\Pi(z) + q\,\Pi(z) + \frac{p}{z}\left[\Pi(z) - \pi_0\right] = \Pi(z)$$

$$\Pi(z) = \frac{p(1-z)\pi_0}{rz^2 + (q-1)z + p} = \frac{p\pi_0}{p - rz} \qquad \frac{\phi(1-z)}{\psi(1-z)}$$

$$\Pi(1) = 1 = \frac{p\pi_0}{p-r} \Rightarrow \pi_0 = 1 - \frac{r}{p}$$

$$0 \le \pi_0 < 1 \Rightarrow 0 < \frac{r}{p} \le 1$$

$$\Pi(z) = \frac{p\pi_0}{p - rz} = \frac{\pi_0}{1 - \frac{r}{p}z} = \frac{A}{1 - \alpha z}, \qquad A = \pi_0, \quad \alpha = \frac{r}{p}$$

Inversion $\qquad \pi_n = \left(\frac{r}{p}\right)^n \pi_0, \quad n = 1, 2, \ldots$

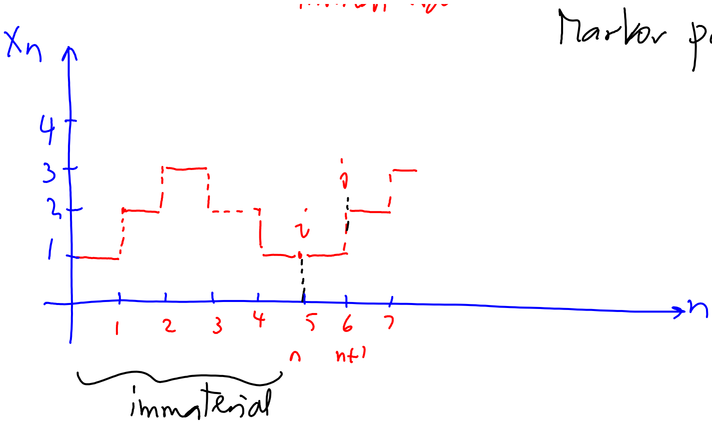
– End of DTMC –

Ch.5 Continuous-time MC (CTMC)

Recall (for DTMC) $\{X_n, n = 0, 1, 2, \ldots\}$ if

$$Pr\{X_{n+1} = j \mid X_n = i, \underbrace{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0}_{\text{immaterial}}\} = Pr\{X_{n+1} = j \mid X_n = i\} = p_{ij}$$

Markov property



Def $\{X(t), t \geq 0\}$ is a CTMC if

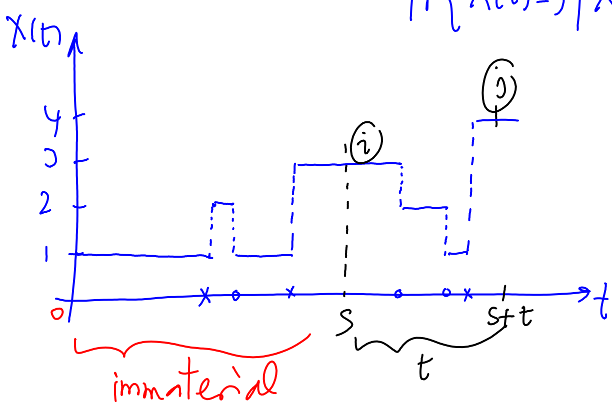
$$\Pr\{X(t+s)=j \mid X(s)=i,\ X(u)=x(u),\ 0 \leq u \leq s\} =$$

$$= \Pr\{X(t+s)=j \mid X(s)=i\} = p_{ij}(t) \qquad (p_{ij}(s,t))$$

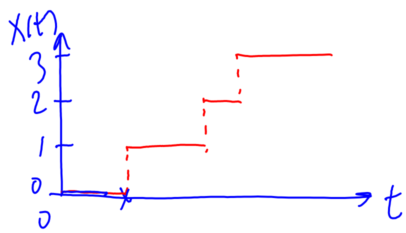
Similar $p_{ij}^{(n)}$

if Time-homogeneous

$$\Pr\{X(t)=j \mid X(0)=i\} = p_{ij}(t)$$

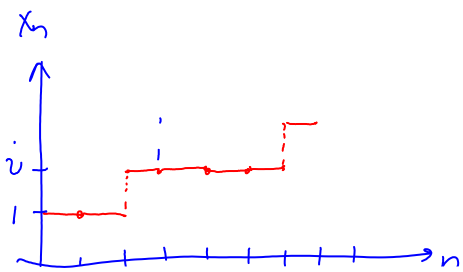


Ex.



Poisson !

Sojourn times

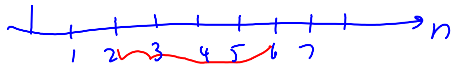


DTMC

N_i : # time units in i

$p_{ii} = \Pr(i \to i)$

$1-p_{ii} = Pr(i \to out)$

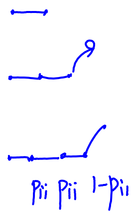


$Pr(N_i=1) = 1-p_{ii}$

$Pr(N_i=2) = p_{ii}(1-p_{ii})$

$Pr(N_i=3) = p_{ii}^2(1-p_{ii})$

$\vdots$



Memorylessness

$Pr(N=m+n \mid N>m) = Pr(N=n)$

$Pr(N=5 \mid N>3) = Pr(N=2)$

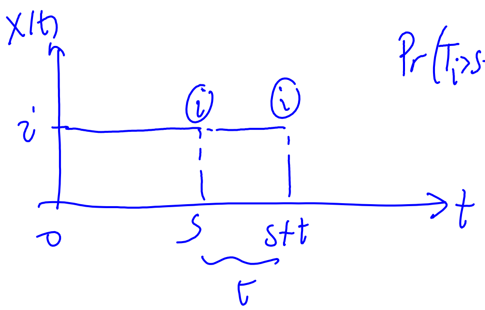
$Pr(N_i=k) = p_{ii}^{k-1}(1-p_{ii}), \quad k=1,2,\ldots$

Show N_i is memoryless

Geometric $\quad E(\text{time in } i) = \dfrac{1}{1-p_{ii}}$

CTMC

T_i: stay time in i



$Pr(T_i > s+t \mid T_i > s) = Pr(T_i > t)$

(from Markov property)

i.e., $\quad f_{T_i}(t) = v_i e^{-v_i t}$, where

v_i: transition $\underline{\text{rate}}$ out of state i per unit time

(Similar to $1-p_{ii}$)

$E(T_i) = \dfrac{1}{v_i}$: Exp time to transition

Summary A CTMC is a SP with the properties that each time it enters a state i,

i) the amount of time it spends in i is expon. with mean $1/v_i$ (dep. rate v_i)

ii) when it leaves i, it enters j with P_{ij} where

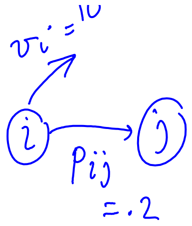
$$P_{ii} = 0, \quad \forall i$$

$$\sum_{j \neq i} p_{ij} = 1 \quad \forall i$$

$i_2 \cdot v_2$
i_3

$v_i = 10$

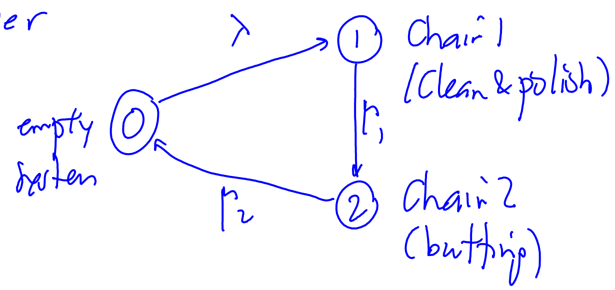
Note.



$q_{ij} = v_i p_{ij}$: transition rate from i to j

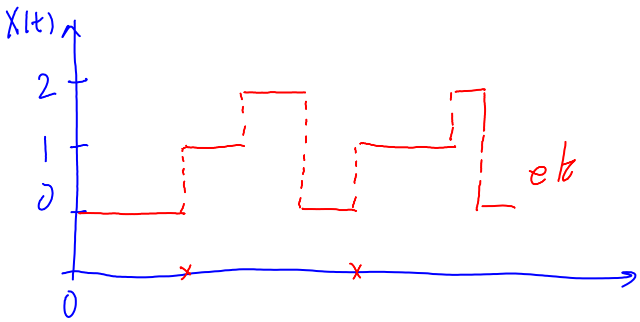
Ex. Shoeshine establishment

One server



. Cust's arrive: Poisson with rate λ

. " enter if both chairs empty



$q_{01} = v_0 p_{01}$

$q_{12} = v_1 p_{12}$

$v_0 = \lambda, \quad v_1 = \mu_1, \quad v_2 = \mu_2$

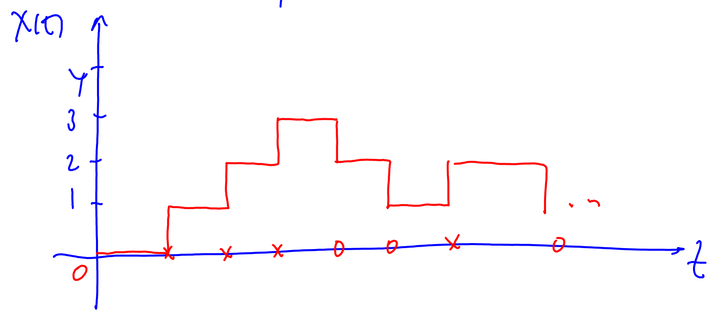
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad Q = [q_{ij}] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} ? & \lambda & 0 \\ 0 & ? & \mu_1 \\ \mu_2 & 0 & ? \end{bmatrix} \end{array}$$

Infinitesimal generator matrix

Note Suppose process was DTMC with λ, μ_1 and μ_2 as prob's

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1-\lambda & \lambda & 0 \\ 0 & 1-\mu_1 & \mu_1 \\ \mu_2 & 0 & 1-\mu_2 \end{bmatrix} \end{array}, \quad P - I = Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu_1 & \mu_1 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} \end{array}$$

Ex. Birth & death process



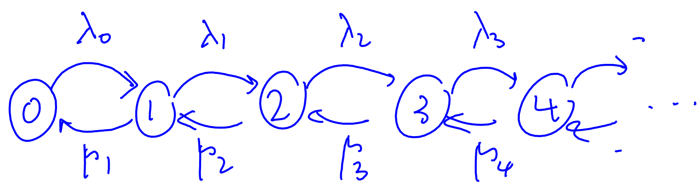
n people in system:

Arrivals $\times$ with rate λ_n (mean $1/\lambda_n$)

Depart $\circ$ " " μ_n (" $1/\mu_n$)

B&D processes are CTMC with transitions

from n $\nearrow n+1$ or $\searrow n-1$

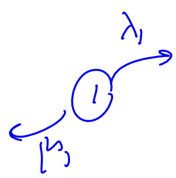


State 0. Time until birth is expon. with rate $v_0 = \lambda_0$

State $i \neq 0$: " " next event (B or D) is expon with rate $v_i = \lambda_i + \mu_i$, $i = 1, 2, \ldots$

X_1: time until birth (exp. λ_1)

Y_1: " " death (" μ_1)



$P_{01} = \Pr(\text{birth at any time in future}) = 1$

$$P_{12} = \Pr(X_1 < Y_1) = \int_0^\infty \Pr(X_1 < Y_1 \mid Y_1 = t)\, \mu_1 e^{-\mu_1 t}\, dt = \frac{\lambda_1}{\lambda_1 + \mu_1}$$

$$P_{10} = \Pr(Y_1 < X_1) = \qquad = \frac{\mu_1}{\lambda_1 + \mu_1}$$

In general $\quad v_0 = \lambda_0$

$$v_i = \lambda_i + \mu_i, \quad i \geq 1$$

$$P_{01} = 1$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \geq 1$$

$\Rightarrow$

$$q_{0,1} = \lambda_0$$

$$q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i \qquad Q = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{bmatrix} ? & \lambda_0 & & & \\ \mu_1 & ? & \lambda_1 & & \\ & \mu_2 & ? & \lambda_2 & \\ & & \ddots & ? & \ddots \\ & & & & \end{bmatrix} \end{array}$$

Ex. Poisson (Pure births)

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} \cdots$$