2013-11-11

Monday, November 11, 2013
12:52 PM

$$\Pi^{(n)} = \Pi^{(0)} P^{(n)}$$

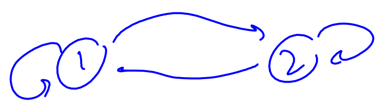
$$\Pi^{(n)} = \Pi^{(n-1)} P$$

$$\Pi^{(n)} - \Pi^{(n-1)} = \Pi^{(n-1)} P - \Pi^{(n-1)}$$
$$= \Pi^{(n-1)} (\underbrace{P - I})$$

Q : rows sum to 0

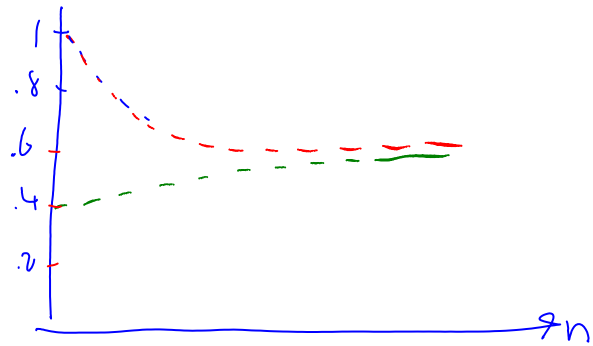
Ex.

$$P = \begin{array}{c|cc} & 1 & 2 \\ \text{Barn } 1 & .8 & .2 \\ \text{fortinos } 2 & .3 & .7 \end{array}$$



$$Q = \begin{bmatrix} -.2 & .2 \\ .3 & -.3 \end{bmatrix}$$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/MarketShare.mw



What can be said about long-run behaviour?

Suppose $\lim_{n\to\infty} P_{ij}^{(n)} = \Pi_j \quad \forall i \in I$

$$\Rightarrow \quad \lim_{n\to\infty} P^{(n)} = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ 0 & \Pi_0 & \Pi_1 & \Pi_2 & \cdots \\ 1 & \Pi_0 & \Pi_1 & \Pi_2 & \cdots \\ 2 & \vdots & \vdots & \vdots & \\ \vdots & & & & \end{array}$$

$$\Rightarrow \quad \lim_{n\to\infty} P^{(n)} = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0 & \pi_0 & \pi_1 & \pi_2 & \cdots \\ 1 & \pi_0 & \pi_1 & \pi_2 & \cdots \\ 2 & \vdots & \vdots & \vdots & \\ \vdots & & & & \end{array}$$

$\{\pi_j\}_{j=0}^{\infty}$: "limiting (steady-state)" prob's

How to find these (if exist?) $\quad \pi = (\pi_0, \pi_1, \ldots)$

$$\pi^{(n)} = \pi^{(n-1)} P$$

$$\lim_{n\to\infty} \pi^{(n)} = \lim_{n\to\infty} \pi^{(n-1)} P$$

$$\boxed{\begin{array}{l} \pi = \pi P \\ \pi e = 1 \end{array}} \qquad (\text{or, } 0 = \pi Q)$$

$$e = (1, 1, \ldots, 1)^T$$

Sol'n gives "stationary" distrib

Ex. Rainfall

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{array} \qquad \begin{pmatrix} .57 \\ .43 \end{pmatrix}$$

$\pi = (\pi_0, \pi_1)$

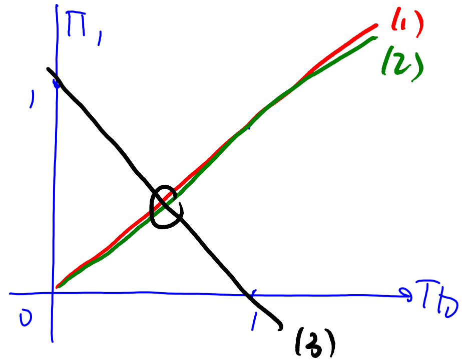
$$(\pi_0, \pi_1) = (\pi_0, \pi_1) \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

(1) $\quad \pi_0 = .7\pi_0 + .4\pi_1 \rightarrow .3\pi_0 = .4\pi_1$:

$\pi_1 = \frac{3}{4}\pi_0 \quad$ (2) $\quad \pi_1 = .3\pi_0 + .6\pi_1 \rightarrow .3\pi_0 = .4\pi_1$

$\pi_1 = \frac{3}{4}\pi_0$ (2) $\pi_1 = .3\pi_0 + .6\pi_1 \rightarrow .3\pi_0 = .4\pi_1$

(3) $\pi_0 + \pi_1 = 1$

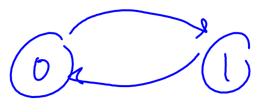


Sol'n: $\pi_0 = .57$

$\pi_1 = .43$

Ex Pathological case

$$P = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$



limiting (?)

$$P^{(n)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & n \text{ odd} \end{cases}$$

$\therefore \lim_{n\to\infty} P^{(n)}$ doesn't exist

Stationary

$$(\pi_0, \pi_1) = (\pi_0, \pi_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\pi e = 1$$

$$\pi_0 = \frac{1}{2}, \quad \pi_1 = \frac{1}{2}$$

Proposition Define

f_j : long-run fraction of time system in j

Then, if π_j exist, we have

$\boxed{f_j = \pi_j}$ (Tijms)

Ex. Above $\underbrace{1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0}\ \ldots$

$\frac{1}{2}$ time in 0

$\frac{1}{2}$ " " 1

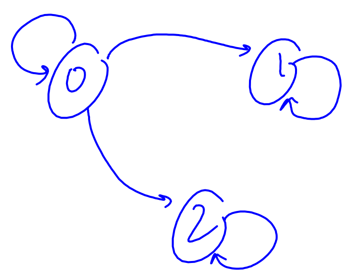
Ex. Three state

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/RedundantEquation.mw

Solution := {x = .2083333333, y = .2222222222, z = .5694444444}

Ex.

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .3 & .5 & .2 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}$$



Stat'ry? $\pi = \pi P$, $\pi e = 1$

$\pi_0 = 0$, $\pi_1 + \pi_2 = 1$

<u>Limiting</u>?

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} 0 & .71 & .29 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Moral</u> Limiting $\Rightarrow$ Stationary

Application (Markov Reward Process): Vehicle fleet insur

Four premium classes $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

Premium $\quad P_1 > P_2 > P_3 > P_4$

exp. $\qquad\qquad$ cheap

Prem. in $n+1$ depends on ① premium in n

② claim history in n

$\uparrow$

P_i

n	$n+1$	
$P_i \to$	P_{i+1}	if no claim in n
$P_i \to$	P_1	if claim in n

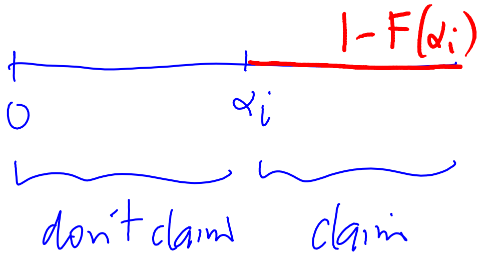
. Total damage in a year is Y $\quad$ pdf f

cdf F

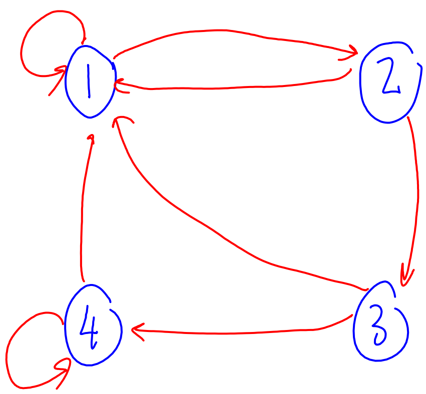
. If claim is made, ins. comp. compensates for damages minus deductible r_i

<u>Problem</u> What claim limits (α_i) minimize

— Yearly Cost



X_n : premium class in year n



$$F(\alpha_i) = \int_0^{\alpha_i} f(y)\,dy$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1-F(\alpha_1) & F(\alpha_1) & 0 & 0 \\ 1-F(\alpha_2) & 0 & F(\alpha_2) & 0 \\ 1-F(\alpha_3) & 0 & 0 & F(\alpha_3) \\ 1-F(\alpha_4) & 0 & 0 & F(\alpha_4) \end{bmatrix} \end{array}$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

Solve $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ $\quad \pi = \pi P$, $\pi e = 1$

LRA cost/yr $\quad g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{j=1}^{4} c(j)\,\pi_j$ (cost rate)

$c(j) = E$ [cost/yr in class j]

$$= P_j + \int_0^{\alpha_j} y f(y)\,dy + \int_{\alpha_j}^{\infty} r_j f(y)\,dy$$

Let

Y is gamma(λ, k) , $E(Y) = \frac{k}{\lambda}$, $Var(Y) = \sigma_Y^2 = \frac{k}{\lambda^2}$

$k \in \mathbb{R}^+$

Define $C_Y^2 = \frac{\sigma_Y^2}{E(Y)^2} = \frac{1}{k}$ Squared Coeff. of variation

$E(Y) = 5{,}000$, $(P_1, P_2, P_3, P_4) = (10000, 7500, 6000, 5000)$

$(r_1, r_2, r_3, r_4) = (1500, 1000, 750, 500)$

		$C_Y^2 = 1$
Optimal	α_1	5,908
	α_2	7,800
	α_3	8,595
	α_4	8,345
		$g = 9{,}058$

b) Transient Solution of MC (via generating functions)

Recall

$$P^{(n)} = P^{(n-1)} P \quad , n = 1, 2, \ldots$$

N states

$$= P^{(0)} P^n \quad , \quad P^{(0)} = I = \begin{bmatrix} P_{00}^{(0)} & \cdots & P_{0N}^{(0)} \\ P_{N0}^{(0)} & \cdots & P_{NN}^{(0)} \end{bmatrix}$$

Define $G(z) = \sum_{n=0}^{\infty} P^{(n)} z^n$, $\left[g(z) = \sum a_n z^n \right]$

matrix   scalar

$$P^{(n)} z^n = P^{(n-1)} P z^n , \quad n = 1, 2, \ldots$$

$$\sum_{n=1}^{\infty} P^{(n)} z^n = \sum_{n=1}^{\infty} P^{(n-1)} P z^n$$

Add & subtract $P^{(0)} z^0$

$$\underbrace{\sum_{n=1}^{\infty} P^{(n)} z^n + P^{(0)} z^0}_{G(z)} - P^{(0)} z^0 = z \sum_{n=1}^{\infty} P^{(n-1)} P z^{n-1}$$

$$G(z) - P^{(0)} z^0 = z \sum_{n=0}^{\infty} P^{(n)} P z^n$$

$$= z \underbrace{\sum_{0}^{\infty} P^{(n)} z^n}\, P$$

$$G(z) - P^{(0)} z^0 = z\, G(z) P$$

$$G(z) - z G(z) P = P^{(0)}$$

$$G(z)(I - zP) = P^{(0)}$$

$$G(z) = p^{(0)} \boxed{(I - zP)^{-1}}$$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$$p^{(n)} = p^{(0)} \cdot P^n$$

Find inv. transform of $(I - zP)^{-1}$

Ex.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{4} & \frac{1}{4} \end{array}$$

$$I - zP = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$(I - zP)^{-1} = \frac{1}{(1-z)(1+\frac{1}{4}z)} \begin{bmatrix} 1 - \frac{1}{4}z & \frac{1}{2}z \\ \frac{3}{4}z & 1 - \frac{1}{2}z \end{bmatrix}$$

Partial fractions $\rightarrow$ invert

$\vdots$

$$P^n = \begin{bmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{bmatrix} + \left(-\frac{1}{4}\right)^n \begin{bmatrix} 2/5 & -2/5 \\ -3/5 & 3/5 \end{bmatrix}$$

$$\lim_{n \to \infty} P^n = \begin{bmatrix} & \\ & \end{bmatrix}$$

c) Classification of states

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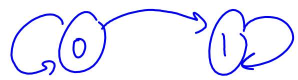
Def j is <u>accessible</u> from i if $\exists n \geq 0$ $\ni$ (such that)

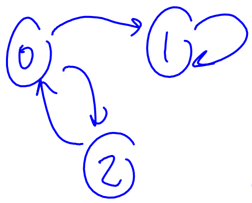
$$p_{ij}^{(n)} > 0 \qquad (i \to j)$$



$0 \to 1$

$1 \not\to 0$

Def i and j <u>communicate</u> if they're accessible from each other $(i \leftrightarrow j)$



$2 \to 1$

$0 \leftrightarrow 2$

$2 \not\leftrightarrow 1$

<u>Proposition</u>. Communication ($\leftrightarrow$) is an <u>equivalence relation</u>, i.e., it satisfies,

(i) $i \leftrightarrow i$ (reflexivity)

(ii) If $i \leftrightarrow j$, then $j \leftrightarrow i$ (Symmetry)

(iii) If $i \leftrightarrow j$ & $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity)

All states can be partitioned into <u>equivalence classes</u>

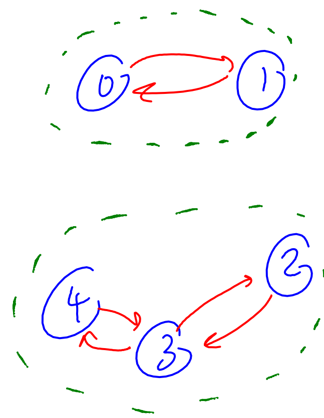
Set of all states that comm. w/ each other

0 1 | 2 3 4

Ex.

comm. w/each other

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|ccc} x & x & & & \\ x & x & & & \\ \hline & & & & \\ & & x & & x \\ & & & & \end{array}\right] \end{array}$$



Two classes $\{0,1\}$, $\{2,3,4\}$

Remark: Kao pp. 172-174
http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/documents/Irreducibility-Algorithm.pdf

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/Reducible-MC-Simpler-N10.mw

Exam: 6 pages (3 sheets) of your summary notes/formulas
Laplace page