2013-11-04

Monday, November 04, 2013

11:26 AM

Ch.4. Markov Chains

Ex. Weather forecasting

W: wet

D: dry WWDDWDDDWD

Independence

$$\Pr\{X_{n+1}=j \mid \underbrace{X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0}_{\text{irrelevant}}\} = \Pr\{X_{n+1}=j\}$$

One-period (Markov) dependency

$$\Pr\{X_{n+1}=j \mid X_n=i, \underbrace{X_{n-1}=i_{n-1}, \ldots, X_0=i_0}_{\text{irrelevant}}\} = \Pr\{X_{n+1}=j \mid X_n=i\}$$

<u>Def</u>. The S.P. $\{X_n, n=0,1,\ldots\}$ is called a discrete-time Markov chain (DTMC), if for each $n=0,1,\ldots$

$$\Pr\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\} = \Pr\{X_{n+1}=j \mid X_n=i\}$$

$$= P_{ij}^{(n,n+1)}$$

for all possible values of $i_0, i_1, \ldots, i_{n-1}, i, j \in I$ (state space)

In the sequel, we consider DTMC with time-homogeneous transition prob's, i.e., time-homogeneous transition prob's, i.e.,

$$\Pr\{X_{n+1}=j \mid X_n=i\} = p_{ij}, \quad i,j\in I$$

independently of time parameter n.

The p_{ij} are one-step transition prob's & they satisfy

$$p_{ij}\geq 0, \quad i,j\in I$$

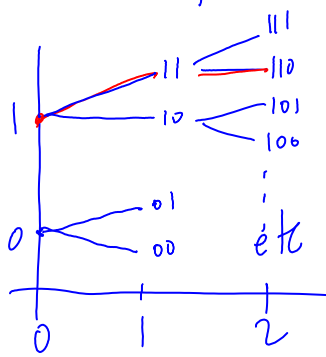
$$\sum_{j\in I} p_{ij}=1, \quad \forall i\in I$$

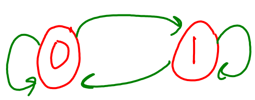
$$\begin{array}{c|cc} & 0\ W & 1\ D \\ \hline 0\ W & p_{00} & p_{01} \\ 1\ D & p_{10} & p_{11} \end{array}$$

$p_{00}+p_{01}=1$

$p_{10}+p_{11}=1$

P: one-step trans. prob. matrix





$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ & & & \end{bmatrix} \end{array}$$

Ex. An (s,S) inventory as a DTMC (periodic review)

Single product

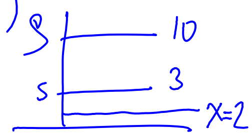
Demands D_n, $n=1,2,\ldots$ are iid with a pdf

$$\phi_k = \Pr(D_n=k), \quad k=0,1,\ldots$$

X_n: inventory position at end of week $n=1,2,\ldots$
(before delivery arrivals, if needed).

x initial invent

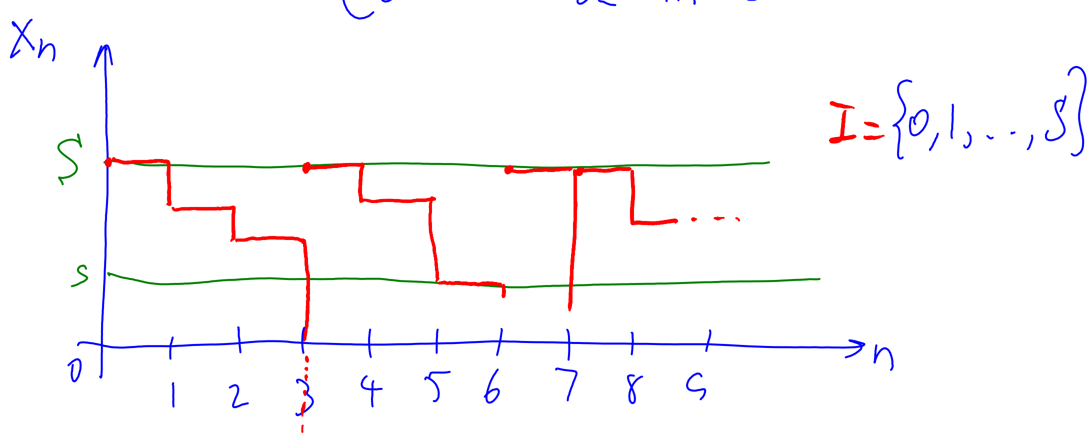
X_0. initial invent needed)



Assume excess demand is lost

Order policy is (s, S) i.e., order Q_n at end of week n

$$Q_n = \begin{cases} S - X_n, & 0 \le X_n < s \\ 0 & s \le X_n \le S \end{cases}$$



Q: Is X_n a MC? i.e., $X_{n+1} \sim X_n$

$$X_{n+1} = \begin{cases} \max(S - D_{n+1}, 0), & 0 \le X_n < s \\ \max(X_n - D_{n+1}, 0), & s \le X_n \le S \end{cases}$$

7 − 2 = 5   3   ↑   10

7 − 11 → 0   7

$X_{n+1} = \phi(X_n)$ ✓ $X_{n+1} = \Psi(X_n, X_{n-1})$ Not 1st order MC

Assume ① $(s, S) = (1, 3)$

② D_n are Poisson with $\lambda = 1$, i.e.,

$$p_k = \Pr(D_n = k) = e^{-\lambda}\frac{\lambda^k}{k!} = \frac{e^{-1}}{k!}, \quad k = 0, 1, \ldots$$

$$P_{00} = \Pr(X_{n+1} = 0 \mid X_n = 0) = \Pr\{\underbrace{\max(3 - D_{n+1}, 0)}_{X_{n+1}} = 0\}$$

$$= \Pr\{D_{n+1} \geq 3\} = 0.080$$

$$P_{10} = \Pr\{X_{n+1} = 0 \mid X_n = 1\} = \Pr\{\max(1 - D_{n+1}, 0) = 0\}$$

$$= \Pr\{D_{n+1} \geq 1\} = 0.632$$

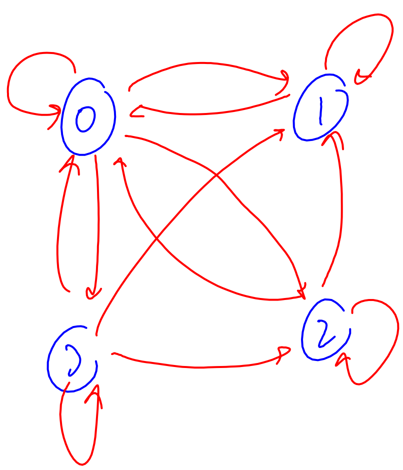
To summarize

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .080 & .184 & .368 & .368 \\ .632 & .368 & 0 & 0 \\ .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \end{bmatrix} \end{array}$$

stationary

$$\left\{ \begin{array}{l} .286 \\ .284 \\ .263 \\ .167 \end{array} \right.$$

later



Ex. Rainfall in Tel Aviv (Gabriel & Neumann '62)

W or D   Dec, Jan, Feb   2437 days (27 yrs)

0   1

$P_{ij} = Pr(X_{n+1} = j \mid X_n = i)$ $P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$

$0 \mid \underbrace{0011000010\cdots}_{2437} 00$

Initial

Simpler $1 \mid 000100101 1$

	W	D	
W	3	3	6
D	3	1	4

$$P = \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} .5 & .5 \\ .75 & .25 \end{bmatrix}$$

Actual data

	W	D	
W	351	687	1,038
D	1,049	350	1,399

$$\to P = \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} .338 & .662 \\ .75 & .25 \end{bmatrix}$$

Ex. Independent trials

let Z be a r.v. with values $0, 1, \ldots, K$

$p_0, p_1, \ldots, p_K$

$X_{n+1} = Z$

$\{X_n, n = 0, 1, \ldots\}$

$Pr\{X_{n+1} = j \mid X_n = i\} = Pr\{X_{n+1} = j\}$

$$\Pr\{X_{n+1}=j \mid X_n = i\} = \Pr\{X_{n+1}=j\}$$

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & \cdots & K \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ K \end{array} & \begin{bmatrix} p_0 & p_1 & \cdots & p_K \\ p_0 & p_1 & & p_K \\ \vdots & & \ddots & \\ p_0 & p_1 & & p_K \end{bmatrix} \end{array}$$

Ex $M/D/1/4$ Queue (shoeshine)

D: 10 min; 4: ↑ system cap

Arrivals Poisson (λ)

3+1 seats $\underbrace{\square\ \square\ \square}_{\text{wait}}\ \boxed{B}$

Service = 10 min

$X(t)$: # in queue at t

$X(t)$ is not even Markovian! We need past history

$$\Pr\{X(t+\delta t)=1 \mid X(t)=1\} = [1-\lambda\delta t + o(\delta t)] \cdot \Pr\{\text{no departure in } (t, t+\delta t)\}$$

$1 \longrightarrow 1$

$t \qquad t+\delta t$

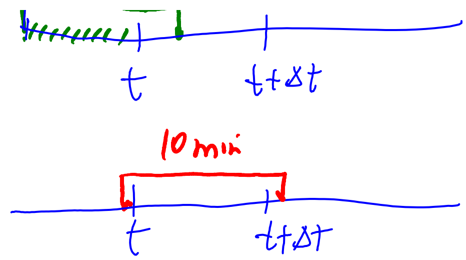
If Service exp. $1-\mu\delta t + o(\delta t)$

If deterministic $1-r_D(t)\delta t + o(\delta t)$

Need not till...

10 min

need past info



complete in (t, t+Δt)

$X_{\Delta t}$ " " "

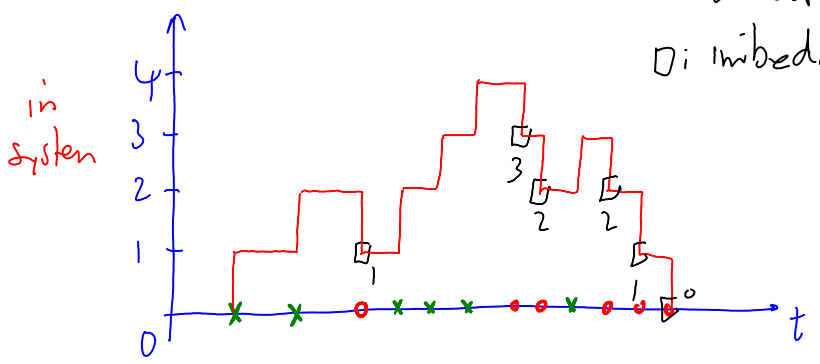
$\{X(t)\}$ not Markovian. How to make it?

X_n : # waiting left behind by nth departure, $n \geq 1$

A_n : # customers arriving during service time of nth customer

embed

D: imbedded process



Colour
Color

A_{n+1}	X_{n+1}
0	0
1	1
2	2
3	3
4	3
5	3

$X_n = 0$

$\boxed{\text{empty (B)}}$

$\boxed{\text{ooo or} \leq \text{(B)}}$ $X_{n+1} = \min(3, A_{n+1})$

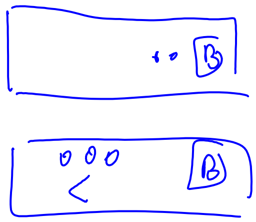
$X_n = 1$

$\boxed{\cdot \text{ (B)}}$

$\boxed{\text{ooo or} \leq \text{(B)}}$ $X_{n+1} = \min(3, 1-1+A_{n+1})$

$= \min(3, A_{n+1})$

$X_n = 2$



$X_{n+1} = \min(3, 2-1+A_{n+1})$

$= \min(3, 1+A_{n+1})$

General

$$X_{n+1} = \begin{cases} \min(3, A_{n+1}), & X_n = 0 \\ \min(3, X_n - 1 + A_{n+1}), & X_n = 1,2,3 \end{cases}$$

$$\Pr(A_{n+1} = j) = \int_0^\infty \Pr(A_{n+1} = j \mid S = s)\, dG(s)$$

$$= e^{-\lambda \cdot 10} \frac{(\lambda \cdot 10)^j}{j!}, \quad j = 0, 1, 2, \ldots$$

$$= a_j$$

$$P_{00} = \Pr(X_{n+1} = 0 \mid X_n = 0) = \Pr(A_{n+1} = 0) = a_0$$

$$P_{01} = \Pr(X_{n+1} = 1 \mid X_n = 0) = \Pr(A_{n+1} = 1) = a_1$$

$$\Rightarrow \quad P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} a_0 & a_0 & a_1 & \geq a_3 \\ a_0 & a_1 & a_2 & \geq a_3 \\ 0 & a_0 & a_1 & \geq a_2 \\ 0 & 0 & a_0 & \geq a_1 \end{bmatrix} \end{array}$$

$\geq a_3 = \sum_{j=3}^{\infty} a_j$

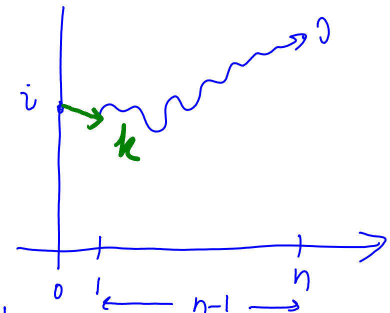
$\geq a_2 = \sum_{j=2}^{\infty} a_j$

$\geq a_1 = \sum_{j=1}^{\infty} a_j$

a) Chapman-Kolmogorov eqn's

$p_{ij} = Pr(X_{n+1}=j \mid X_n=i)$. 1-step

$p_{ij}^{(n)} = Pr(X_{m+n}=j \mid X_m=i)$, $n,i,j \geq 0$ n-step



$$p_{ij}^{(n)} = Pr(X_n=j \mid X_0=i)$$

$$= \sum_{k\in I} Pr(X_n=j \mid X_1=k, \cancel{X_0=i}) \cdot Pr(X_1=k \mid X_0=i)$$

$$= \sum_{k\in I} p_{kj}^{(n-1)} p_{ik} = \sum_{k\in I} p_{ik}\, p_{kj}^{(n-1)}$$

ij-th element of $P^{(n)}$

$$\Rightarrow P^{(2)} = P\cdot P = P^2$$

$$P^{(3)} = P\cdot P^{(2)} = P^3$$

$$\boxed{P^{(n)} = P^n}$$

Ex Rainfall

Let $P = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \end{array}$

evalm(P^200); [[0.5714285714,0.4285714286],[0.5714285714,0.4285714286]]

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .57142 & .4285 \\ 1 & .57142 & .4285 \end{array}$$

Unconditional Prob.

$\Pi_j^{(n)} = Pr(X_n = j)$, $\Pi^{(n)} = (\Pi_0^{(n)}, \Pi_1^{(n)}, \ldots)$

Initial $\Pi^{(0)} = (\Pi_0^{(0)}, \Pi_1^{(0)}, \ldots)$

$$\Pi_j^{(n)} = Pr(X_n = j) = \sum_{i\in I} \underbrace{Pr(X_n = j \mid X_0 = i)}_{P_{ij}^{(n)}} \, Pr(X_0 = i)$$

$$Pr(X_0 = i) \downarrow \Pi_i^{(0)}$$

$$= \sum_{i\in I} \Pi_i^{(0)} P_{ij}^{(n)}$$

$$\Pi^{(n)} = \Pi^{(0)} P^{(n)}$$

Or, Conditioning on last posn

$$\boxed{\Pi^{(n)} = \Pi^{(n-1)} P}$$

(Next wk

$\Pi = \Pi P$

$\Pi e = 1$)