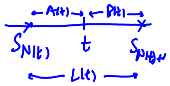
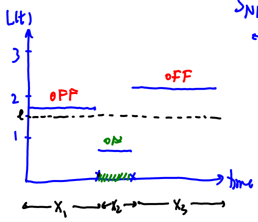
2013-10-28

Monday, October 28, 2013
12:18 PM

Now consider $L(t) = A(t) + B(t)$ | $F(x) = Pr\{X \le x\}$





$\{ON \text{ at } t\} \Leftrightarrow \{L(t) \le \ell\}$

$ON = \begin{cases} X, & X \le \ell \\ 0, & X > \ell \end{cases}$

\ell

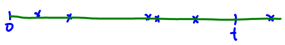
$E(ON) = \int_0^\ell x\,dF(x) + \int_\ell^\infty 0\,dF(x)$

$W_L(\ell) = Pr\{L \le \ell\} = \lim_{t\to\infty} Pr\{L(t) \le \ell\} = \frac{E(ON)}{\mu}$ ← ARP

$W_L(\ell) = \frac{\int_0^\ell x\,dF(x)}{\mu}$

$\therefore\ w_L(\ell) = \frac{1}{\mu}\ell f(\ell)$: Poisson. $w_L(\ell) = \lambda^2 \ell e^{-\lambda\ell}$: Erlang(2,λ)

$E(L) = \mu + \frac{\sigma^2}{\mu} > E(X) = \mu$: Inspection "paradox"

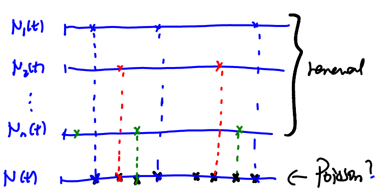


Application: Why does Poisson occur naturally in practice?

Superposition of a large # of renewal processes

$\{N_k(t), t \ge 0\}$, $k = 1, 2, \ldots, n$ Independent

$N(t) = \sum_{k=1}^{n} N_k(t)$: Superposition process

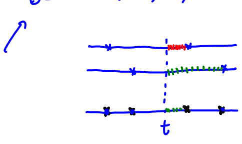


Let $\mu_k = \int_0^\infty x\,dF_k(x)$: mean interarrival time for process $k = 1, \ldots, n$

In steady-state $(t \to \infty)$

Let B_k = forward recurrence time (FRT) of process $k = 1, \ldots, n$

$B = \min(B_1, \ldots, B_n)$: FRT until next event in $\{N(t)\}$



Suppose n gets large & in each process k, renewals occur rarely. More precisely, arrival rates $\frac{1}{\mu_k}$ get small ∋

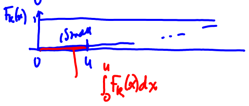
$$\frac{1}{\mu_1} + \frac{1}{\mu_2} + \cdots + \frac{1}{\mu_n} \longrightarrow \lambda \quad \text{as } n \to \infty$$

Now, $Pr\{B > u\} = Pr\{B_1 > u\} \cdot Pr\{B_2 > u\} \cdots Pr\{B_n > u\}$

$Pr\{B_k \le u\} = \frac{1}{\mu_k}\int_0^u [1 - F_k(x)]\,dx$ (last week)

Fix u. For large n, μ_k is large so that for

any k,



$F_k(x) \cong 0$ for $0 \le x \le u$

$Pr\{B_k \le u\} = \frac{1}{\mu_k} \cdot u$. So,

$Pr\{B > u\} \cong \left(1-\frac{u}{\mu_1}\right)\left(1-\frac{u}{\mu_2}\right)\cdots\left(1-\frac{u}{\mu_n}\right)$

$\simeq 1-\left(\frac{u}{\mu_1}+\frac{u}{\mu_2}+\cdots+\frac{u}{\mu_n}\right)$

$\cong 1-u\left(\frac{1}{\mu_1}+\cdots+\frac{1}{\mu_n}\right)$

$\approx 1-u\lambda = e^{-\lambda u}$

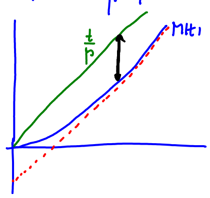
$e^{-\lambda u} \simeq 1-u\lambda$

$\therefore$ $N(t)$ is Poisson $\square$

$\left[h_B(b) = \lambda e^{-\lambda b} \right]$ ↑ PRT

Recall. ERT : $\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$ ($\phi(t)$)

$M(t) \overset{?}{\to} \frac{t}{\mu}$ No!

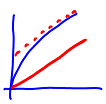


mea culpa

Corollary If $E(X^2) < \infty$, then

$M(t) - \frac{t}{\mu} \to \left(\frac{E(X^2)}{2\mu^2} - 1\right) \to \frac{\sigma^2 - \mu^2}{2\mu^2}$ as $t \to \infty$

$\sigma^2 = Var(X)$, $\mu = E(X)$



Proof. Uses $B(t)$ & Wald's eqn (didn't do)

Ex. $X \sim Erl(2, \lambda)$, $\lambda = .25$, $\mu = \frac{2}{.25} = 8$

$E(X^2) = \frac{6}{\lambda^2} = 96$

$\mu^2 = 64$

mgf $\phi(t) = \left(1-\frac{t}{\lambda}\right)^{-n}$

$\phi''(0) = \frac{6}{\lambda^2}$

Exact: $M(t) = \frac{1}{2}\lambda t - \frac{1}{4}\left[1 - e^{-2\lambda t}\right]$

$t = 50$: $M(50) = 6.00$

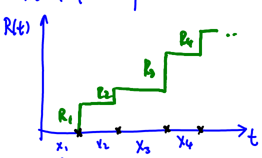
with wrong idea, $\lim_{t\to\infty} M(t) \ne \frac{t}{\mu}$. $M(50) \cong 50(.125) = 6.25$

With corollary, $\lim_{t\to\infty}\left\{M(t) - \frac{t}{\mu}\right\} = \frac{E(X^2)}{2\mu^2} - 1$

$M(50) \simeq \frac{50}{8} - \frac{1}{4} = 6.25 - .25 = 6.$ (exact)

c) Renewal reward Processes

Extension of compound Poisson



$\{N(t), t \ge 0\}$ renewal

X_n, $n \ge 1$ cdf F

R_n: "reward" at nth renewal

Assume (X_n, R_n) are iid

Assume (X_n, R_n) are iid (may depend)

$$R(t) = \sum_{n=1}^{N(t)} R_n \quad : \text{total reward}$$

Theorem (Renewal reward Theorem: RRT)

If $E(R) < \infty$ and $\mu = E(X) < \infty$, then

(i) w.p.1 $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)}$

(ii) $\lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E(R)}{E(X)}$

$\frac{R(1)}{1}, \frac{R(2)}{2}, \dots$

$Z_n \to z$ wp1 if $\Pr\{\lim_{n\to\infty} Z_n = z\} = 1$

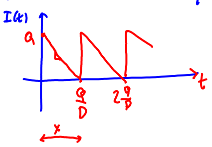
Proof (i)

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t} \to E(R) \cdot \frac{1}{E(X)} = \frac{E(R)}{E(X)}$$

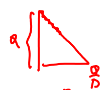
Law of large #s in stats; law of large #s for renewal process

(ii) Wald's eqn needed

Ex. Economic order quantity (EOQ)



D: demand rate
Q: quantity ordered
K
h



$E(R) = K + \frac{1}{2} h \frac{Q^2}{D}$ $\qquad h \frac{1}{2} Q \cdot \frac{Q}{D}$

$E(\text{cycle}) = \frac{Q}{D}$

$$C(Q) = AC(Q) = \frac{E(R)}{E(\text{cycle})} = \frac{K + \frac{1}{2} h \frac{Q^2}{D}}{\frac{Q}{D}} = K\frac{D}{Q} + \frac{1}{2} h Q$$

$$Q^* = \sqrt{\frac{2KD}{h}}$$

$C'(Q) = -\frac{KD}{Q^2} + \frac{1}{2} h = 0$

$\frac{KD}{Q^2} = \frac{1}{2} h$

ceteris paribus

$2KD = Q^2 h$

$Q^* = \sqrt{\frac{2KD}{h}}$

$C^*(Q^*) = \sqrt{2KDh}$

[#12: frequency of order: 4 weeks between order $\frac{1}{4} = 0.25$ orders/wk]

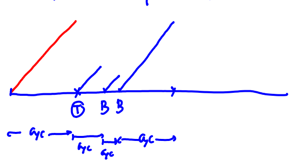
Ex. Car buying (age replacement)

Car's lifetime X has F, f

Replace if Breakdown "B" OR "T"

C_1: cost of a new car

C_2: additional cost of breakdown

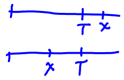


Cycle: Buying a new car either due to "B" or "T"

$$E(C) = \int_0^\infty E(C \mid X = x)\, dF(x) \quad : \quad E(\text{Cost})$$

$E(C|X=x) = \{ C_1 \quad x > T$

$E(C|X=x)=\begin{cases} C_1 & x>T \\ C_1+C_2 & x\le T\end{cases}$



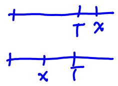
$E(C)=\int_T^\infty C_1\,dF(x)+\int_0^T (C_1+C_2)\,dF(x)$

$E(C)=C_1+C_2F(T)$

L: cycle length

$E(L)=\int_0^\infty E(L|X=x)\,dF(x)$

$E(L|X=x)=\begin{cases} T & x>T \\ x & x\le T\end{cases}$



$E(L)=\int_T^\infty T\,dF(x)+\int_0^T x\,dF(x)$

$\quad=(1-F(T))T+\int_0^T x\,dF(x)$

g(T): long-run avg cost

$g(T)=\frac{E(C)}{E(L)}=\frac{C_1+C_2F(T)}{T(1-F(T))+\int_0^T xf(x)dx}$

$g'(T)=0$

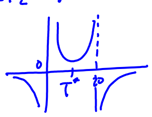
Hope that $g''(T)>0$

$g'(T)=0$

If $X\sim U[0,10]$ and $C_1=\$3\,(10K)$ \$30,000

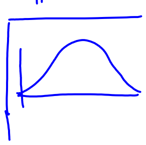
$C_2=\$\frac{1}{2}(10K)$: \$5000

$g(T)=\frac{60+T}{20T-T^2}$



$\rightarrow T^*=9.25$ yrs

$g(T^*)=.69\,(10K)=\$6,900$



$e^{-x_i^2/2h}\cdots$

MT Nov. 15 (F) 10am-1pm

Dec 2 (M) last class (may start later)

Dec. 16 (M) Final?

Hw2 due Dec. 11?