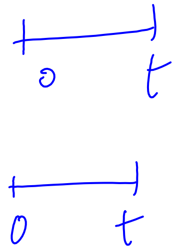
2013-10-18

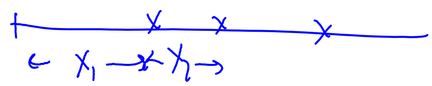
Friday, October 18, 2013
12:30 PM

$P_n(t) = Pr\{N(t) = n\}$ Poisson

$P_n(t) = Pr\{n$ in system at $t\}$



Ch. 3. Renewal Theory



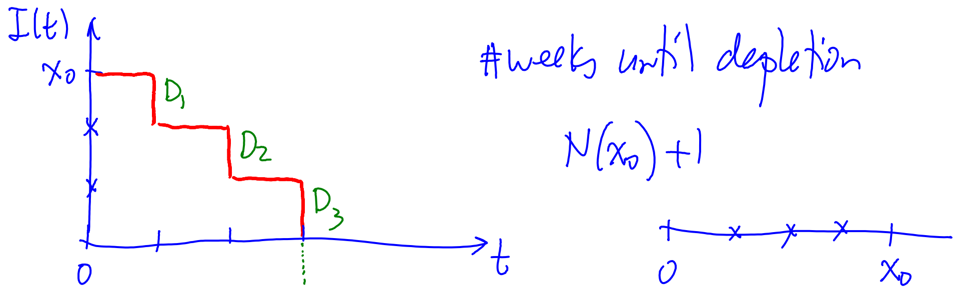
Ex. ① Customers arriving, $N(t)$.

② Light bulbs burning & replaced

$N(t)$

③ Periodic review inventory

D_i: demand in wk i



#weeks until depletion

$N(x_0)+1$

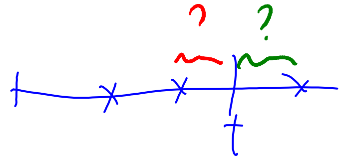
IF Poisson

① Distribution of $N(t)$ — Poisson

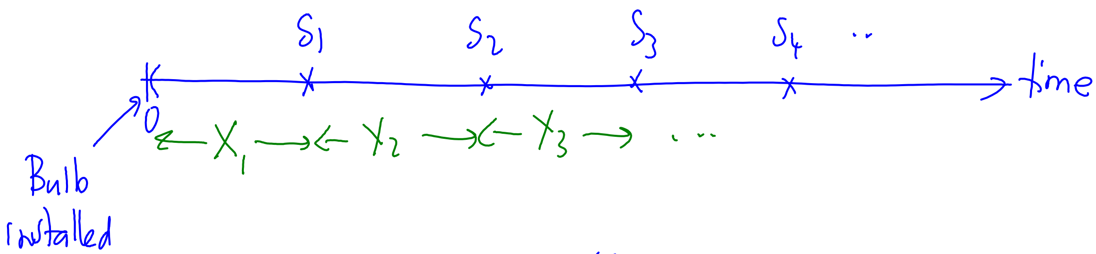
② $E(N(t))$ — λt

③ Time since last occurrence (backward recurrence)

& " until next " (forward r.)



(a) Preliminaries

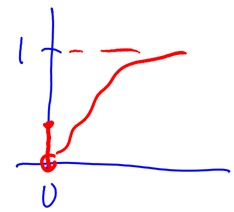


$N(t)$. # renewals by time t

$S_0 = 0$

$S_n = X_1 + \cdots + X_n \; , \; n \geq 1$

$\{X_n ; n \geq 1\}$: iid with F : $F(0) = Pr(X_n = 0) < 1$

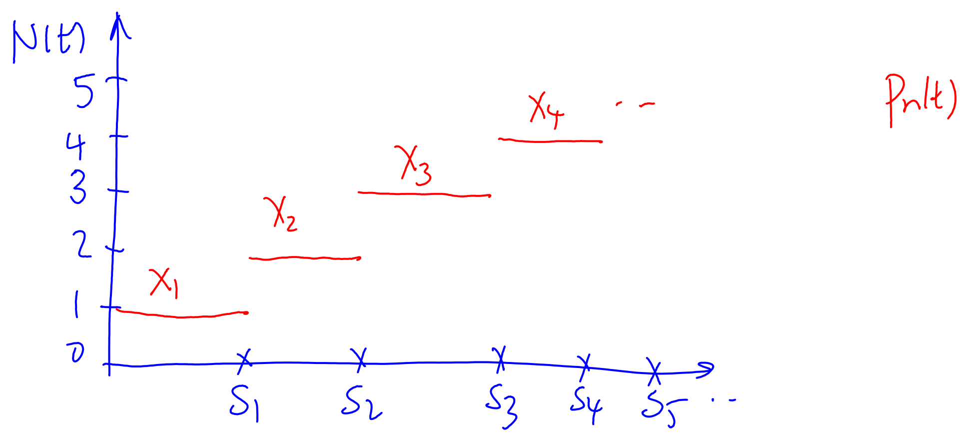


$\mu = E(X_n) = \int_0^\infty x \underbrace{dF(x)}$

Riemann-Stieltjes

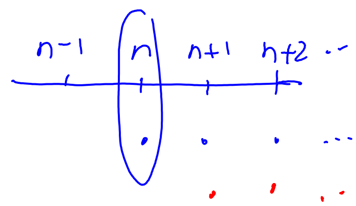
$= \int_0^\infty x f(x) dx$: mean time between events or, to failure

$\frac{1}{\mu}$: rate of process $\quad (\mu = 0.5 \, , \; \frac{1}{\mu} = 2)$



• Q: What's distribution of $N(t)$? $\left[\begin{array}{l} \text{If Poisson} \\ P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{array} \right]$

We know $\{S_n \le t\} \Leftrightarrow \{N(t) \ge n\}$



$$p_n(t) = Pr\{N(t)=n\} = Pr\{N(t) \ge n\} - Pr\{N(t) \ge n+1\}$$

$$= Pr\{S_n \le t\} - Pr\{S_{n+1} \le t\}$$

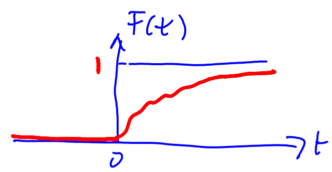
$$p_n(t) = F_n(t) - F_{n+1}(t)$$

where $F_n(t) = F_1(t) \circledast F_{n-1}(t)$

$$= \int_0^{\infty} F_1(t-u) \cdot f_{n-1}(u)\, du \quad : n\text{-fold convolution}$$

cdf - known

$F_1(t)$



$$F_2(t) = \int_0^{\cancel{\infty}\, t} F_1(t-u) \cdot f_1(u)\, du$$

F(t-u)=0 if u>t

$$F_3(t) = \int_0^{\cancel{\infty}\, t} F_1(t-u) f_2(u)\, du, \quad f_2(u) = F_2'(u)$$

$\vdots$

Exercise. Show that if $F(t) = 1 - e^{-\lambda t}$, then

$$p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

- Renewal function $M(t) = E[N(t)]$

$$E(X) = \sum x p(x)$$

<u>Proposition</u> 1 $\quad M(t) = \sum_{n=1}^{\infty} F_n(t)$

Proof $\quad M(t) = E[N(t)] = \sum_{n=0}^{\infty} n\, Pr\{N(t)=n\}$

Proof $M(t) = E[N(t)] = \sum_{n=1}^{\infty} n \Pr\{N(t)=n\}$

$$= \sum_{n=1}^{\infty} n \{F_n(t) - F_{n+1}(t)\}$$

$$= F_1 - \cancel{F_2}$$
$$2\cancel{F_2} - 2\cancel{F_3}$$
$$3\cancel{F_3} - 3F_4$$

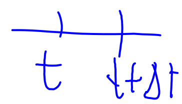
✓ Telescopic cancellations

$$= F_1 + F_2 + F_3 + \cdots$$

$$= \sum_{n=1}^{\infty} F_n(t) \qquad \blacksquare$$

Differentiate $M'(t) = m(t) = \sum_{n=1}^{\infty} f_n(t)$: renewal density

$$\boxed{m(t)\Delta t \simeq \sum_{n=1}^{\infty} f_n(t)\Delta t}$$



Also, $m(t) \to M(t)$?

$$M(t) = \int_0^t m(u)\,du \quad , \quad M(0) = 0$$

Proposition 2 There's a 1-1 correspondance between $F(t)$ and $M(t)$

Proof LT $\tilde{f}(s) = \int_0^{\infty} e^{-st} f(t)\,dt$

$$\tilde{F}(s) = \int_0^{\infty} e^{-st} F(t)\,dt$$

$$\tilde{M}(s)=\int_0^\infty e^{-st}M(t)\,dt$$

$$\tilde{m}(s)=\int_0^\infty e^{-st}m(t)\,dt$$

Recall $m(t)=\sum_{n=1}^{\infty} f_n(t)$

$$\tilde{m}(s)=\sum_{n=1}^{\infty}\tilde{f}_n(s)=\sum_{n=1}^{\infty}\left[\tilde{f}(s)\right]^n$$

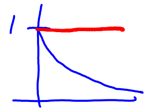
$$\sum_{n=1}^{\infty}x^n=x+x^2+x^3$$

$$=x(1+x+x^2+\cdots)$$

$$=x\,\frac{1}{1-x}$$

$$=\frac{\tilde{f}(s)}{1-\tilde{f}(s)}$$

$0<\tilde{f}(s)<1$



① $\int_0^\infty f(t)\,dt=1$

② $\int_0^\infty e^{-st}f(t)\,dt<\int_0^\infty 1\,f(t)\,dt=1$

$$\Rightarrow\ \tilde{f}(s)=\frac{\tilde{m}(s)}{1+\tilde{m}(s)}$$

Ex. let $f(t)=te^{-t},\ t\geq 0$, Erlang$(\overset{n}{2},\overset{\lambda}{1})$ $\left(\frac{\lambda}{\lambda+s}\right)^n$

$E(X)=\frac{n}{\lambda}=2$, $Var(X)=\frac{n}{\lambda^2}=2=\sigma^2$

$$\tilde{f}(s)=\int_0^\infty e^{-st}f(t)\,dt=\frac{1}{(1+s)^2}$$

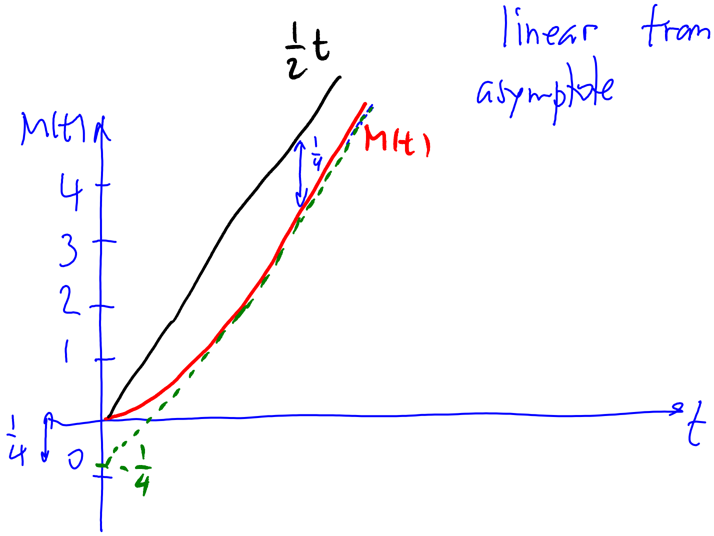
$$\tilde{F}(s)=\frac{1}{s}\tilde{f}(s)=\frac{1}{s(1+s)^2}$$

$$\tilde{m}(s)=\frac{\tilde{f}(s)}{1-\tilde{f}(s)}=\frac{1}{s(s+2)}$$

Inverting $m(t)=\frac{1}{2}-\frac{1}{2}e^{-2t}$, $M(0)=0$

$$M(t)=\int_0^t m(u)\,du=\frac{1}{2}t+\frac{1}{4}e^{-2t}-\frac{1}{4}$$

linear asymptote   transient   bias



$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{2} = \frac{1}{\mu}$$ Elementary renewal Thm

$$\lim_{t\to\infty} m(t) = \frac{1}{2} = \frac{1}{\mu}$$

$$\lim_{t\to\infty} \left\{ M(t) - \underbrace{\tfrac{1}{2}t}_{\text{lin. asym}} \right\} = \lim \left\{ \frac{1}{4} e^{-2t} - \frac{1}{4} \right\} = -\frac{1}{4}$$ bias

In general, $$\lim_{t\to\infty} \left\{ M(t) - \frac{t}{\mu} \right\} = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

Propos. 3 $M(t) < \infty \quad \forall t \in [0, \infty)$

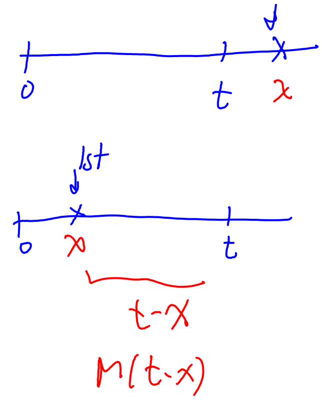
(b) Renewal equation

$$M(t) = E[N(t)] = \int_0^\infty \underbrace{E[N(t) \mid X_1 = x]}_{\text{condition on the occurrence of 1st renewal}} \, \overbrace{dF(x)}^{f(x)dx}$$

Fix t

1st

$$E[N(t) \mid X_1 = x] = \begin{cases} 0, & x > t \\ 1 + M(t-x), & x \le t \end{cases}$$



"Renewal argument"

$$\therefore \quad M(t) = \int_0^t [1 + M(t-x)]\, dF(x) + \cancel{\int_t^\infty 0 \cdot dF(x)}$$

$$= \int_0^t \underbrace{dF(x)}_{f(x)dx} + \int_0^t M(t-x)\, dF(x)$$

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x)$$

Integral eq'n of renewal Theory

$$= F(t) + \int_0^t M(t-x) f(x)\, dx$$

$$\Rightarrow \quad m(t) = f(t) + \int_0^t m(t-x) f(x)\, dx$$

$$\tilde{m} = \tilde{f} + \tilde{m}\tilde{f} \quad \Rightarrow \quad \tilde{m} = \frac{\tilde{f}}{1 - \tilde{f}}$$

Generalization (Solution of "renewal-type" eqn)

- If $\quad g(t) = \underset{\text{any}}{h(t)} + \int_0^t g(t-x)\, d\underset{\text{cdf}}{F(x)}$,

then (Sol'n is) $\quad g(t) = h(t) + \int_0^t h(t-x)\, dM(x)$

where $\quad M(t) = \sum F_n(t)$, as before

Proof $\quad \tilde g = \tilde h + \tilde g \tilde f$

$$\Rightarrow \tilde g = \frac{\tilde h}{1-\tilde f} = \tilde h\left[1+\frac{\tilde f}{1-\tilde f}\right]$$

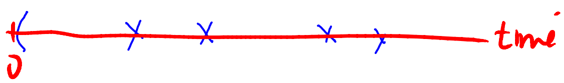
$$= \tilde h + \tilde h \underbrace{\left(\frac{\tilde f}{1-\tilde f}\right)}_{\tilde m}$$

$$\Rightarrow \tilde g = \tilde h + \tilde h \tilde m$$

Invert: $\quad g(t) = h(t) + \int_0^t h(t-x)\, \overbrace{dM(x)}^{m(x)dx}$

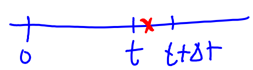
Ex. Equipment replacement $\quad$ (hazard rate)

$$r(t) = \frac{f(t)}{\bar F(t)}$$



lifetimes are iid with $f(t)$

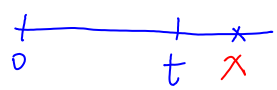
$r(t)\Delta t \simeq \Pr\{\ \ |\cdot\}$



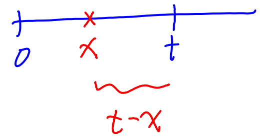
$g(t)$ = failure rate of equipment at time t $\quad$ (Find $g(t)$)

$$g(t) = \int_0^\infty \underbrace{g(t\,|\,X_1=x)}\, f(x)\,dx$$

$$g(t\,|\,X_1=x) = \begin{cases} \frac{f(t)}{\bar F(t)}, & x>t \\ & \end{cases}$$



$\left(g(t-x), \quad x \le t \right.$



$\therefore$

$$g(t) = \int_t^\infty \frac{f(t)}{\bar{F}(t)} f(x)\,dx + \int_0^t g(t-x) f(x)\,dx$$

$$= \cdots$$

$$= -$$

$$g(t) = f(t) + \int_0^t g(t-x) f(x)\,dx$$

Sol'n $\quad g(t) = f(t) + \int_0^t f(t-x)\,m(x)\,dx$

Ex. $\quad X \sim \exp(\lambda) \qquad M(t) = \lambda t \qquad f(t) = \lambda e^{-\lambda t}$

$\qquad m(t) = \lambda \qquad f(t-x) = \lambda e^{-\lambda(t-x)}$

$$\Rightarrow \quad g(t) = \lambda e^{-\lambda t} + \int_0^t \lambda e^{-\lambda(t-x)} \lambda\,dx$$

$$= \lambda$$

Ex. $\quad X \sim \text{Erlang}\,(\overset{n}{2}, \overset{\lambda}{3}) \quad , \quad f(t) = 9te^{-3t} \quad , \quad \mu = \frac{n}{\lambda} = \frac{2}{3}$

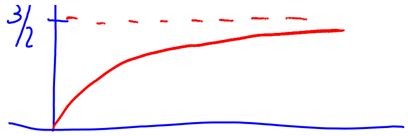
$$f(t-x) = 9(t-x)\,e^{-3(t-x)}$$

$$m(x) = \frac{3}{2} - \frac{3}{2} e^{-6x}$$

$$g(t) = f(t) + \int_0^t f(t-x)\,m(x)\,dx$$

$$= \frac{3}{2} - \frac{3}{2} e^{-6x}$$

$$\lim_{t\to\infty} g(t) = \frac{3}{2} = \frac{1}{p}$$



For 1st one (item) only, $r(t) = \dfrac{f(t)}{\bar{F}(t)} = \dfrac{9t}{1+3t} \longrightarrow 3$