2013-10-07

Monday, October 07, 2013
12:12 PM

$$E(X) = \sum E(X \mid N=n) \Pr(N=n)$$

No class on 14th

Make-up " " 18th? Time? 1-4?

$$E[W(T)] = E\left[\sum_{i=1}^{N(T)} (T - S_i)\right] = \sum_{n=0}^{\infty} \underbrace{E\left[\sum_{i=1}^{N(T)} (T-S_i) \,\Big|\, N(T)=n\right]} \cdot \Pr[N(T)=n]$$

Inside summation:

$$E\left[\sum_{i=1}^{N(T)} (T-S_i) \,\Big|\, N(T)=n\right] = nT - E\left[\sum_{i=1}^{N(T)} S_i \,\Big|\, N(T)=n\right]$$

$$= nT - E\left[\sum_{i=1}^{n} U_{(i)}\right] = nT - E\left[\sum_{i=1}^{n} U_i\right], \quad U_i \sim unif(0,T)$$

$0 \quad \frac{1}{2}T \quad T$

$$= nT - \left(\frac{1}{2}T\right)n = \frac{1}{2}nT$$

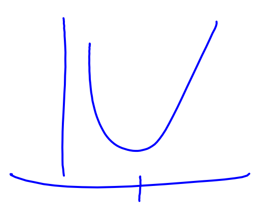
$$\Rightarrow E[W(T)] = \sum_{n=0}^{\infty} \frac{1}{2} nT \cdot \underbrace{\Pr[N(T)=n]}$$

$$= \frac{1}{2}\lambda T^2 \qquad \left[\frac{u}{t} \cdot t^2\right] = [u \cdot t]$$

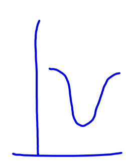
$$\therefore E[\text{Backorder cost}] = \frac{1}{2}\hat{\pi}\lambda T^2 \qquad \left[\frac{\$}{u \cdot t} \cdot \frac{u}{t} \cdot t^2\right] = \$$$

$$\Rightarrow RRT: \; AC(T) = \frac{K + c\lambda T + \frac{1}{2}\hat{\pi}\lambda T^2}{T} \qquad \left(\frac{\$}{t}\right)$$

$$C(T) = \frac{K}{T} + c\lambda + \frac{1}{2}\hat{\pi}\lambda T$$



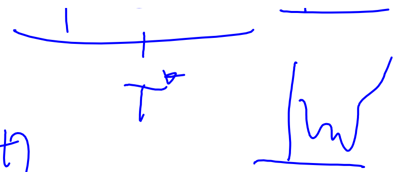
T^*



$* \sqrt{2K \ldots} \qquad \sqrt{\ldots *}$

$$T^* = \sqrt{\frac{2K}{\hat{\pi}\lambda}}$$

$$\sqrt{\frac{\$}{\frac{\$}{u \cdot t} \cdot \frac{u}{t}}} = \sqrt{t^2} = [t]$$



$$C^* = \sqrt{2\lambda\hat{\pi}K} + c\lambda$$

ceteris paribus

$$C'(T) = 0, \quad C''(T) > 0$$

d) Non-homogeneous Poisson process

Ordinary Poisson

Rate $\lambda > 0$

(i) $N(0) = 0$

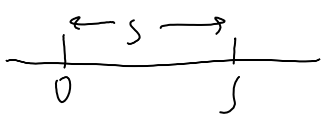
(ii) Ind't & stat. increm

(iii) $Pr\{N(h) = 1\} = \lambda h + o(h)$

(iv) $Pr\{N(h) \geq 2\} = o(h)$

$E(N(t)) = \lambda t = M(t)$

$P_n(s) = Pr\{N(s) = n\} = e^{-\lambda s} \frac{(\lambda s)^n}{n!}$



Non-homogeneous Poisson

Rate $\lambda(t) \geq 0$

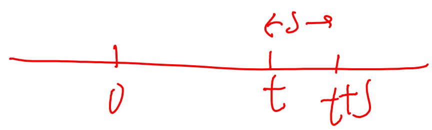
(i) $N(0) = 0$

(ii) Ind't increment

(iii) $Pr\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$

(iv) $Pr\{N(t+h) - N(t) \geq 2\} = o(h)$

$M(t) = \int_0^t \lambda(u)\,du$



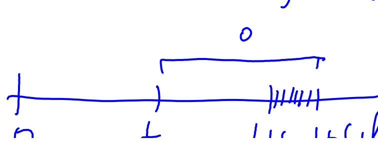
Theorem:

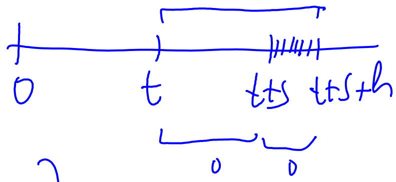
$$P_n(s) = P_n(t,s) = Pr\{N(t+s) - N(t) = n\}$$

$$= e^{-[M(t+s) - M(t)]} \frac{\{M(t+s) - M(t)\}^n}{n!}$$

Proof (Sketch)

$P_n(s) = Pr\{N(t+s) - N(t) = n\}$ for fixed t





$$P_0(s+h) = \Pr\{N(t+s+h)-N(t)=0\}$$

$$= P_0(s)\{1-\lambda(t+s)\cdot h + o(h)\}$$

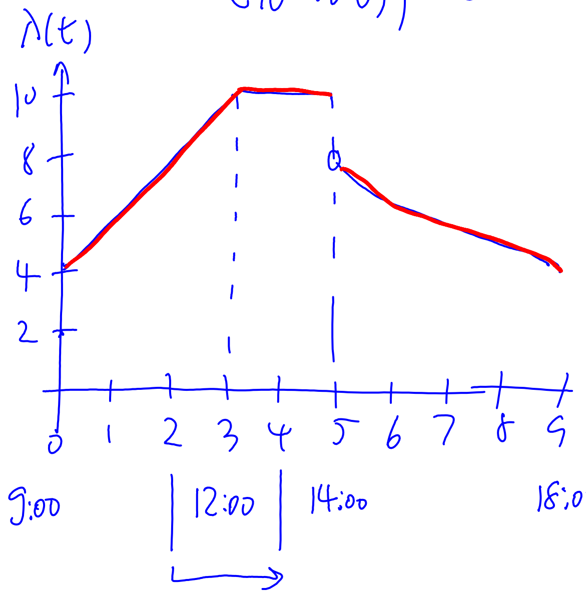
$$\Rightarrow \left.\begin{array}{l} P_0'(s) = -\lambda(t+s)\cdot P_0(s) \\ P_0(0)=1 \end{array}\right\} \Rightarrow P_0(s) = e^{-[M(t+s)-M(t)]}$$

$$\Rightarrow P_n'(s) = -\lambda(t+s)P_n(s) + \lambda(t+s)P_{n-1}(s), \quad n \geq 1 \qquad \text{induction}$$

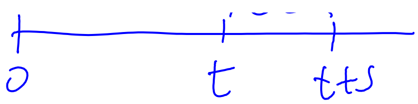
h2

Ex.: Rush hour business

$$\lambda(t) = \begin{cases} 4+2t, & 0 \leq t \leq 3 \\ 10, & 3 \leq t \leq 5 \\ 10-(t-3), & 5 < t \leq 9 \end{cases}$$



Need to use correct formula



$Pr\{15 \text{ customers between } 11{:}00 \text{ and } 13{:}00\} = ?$

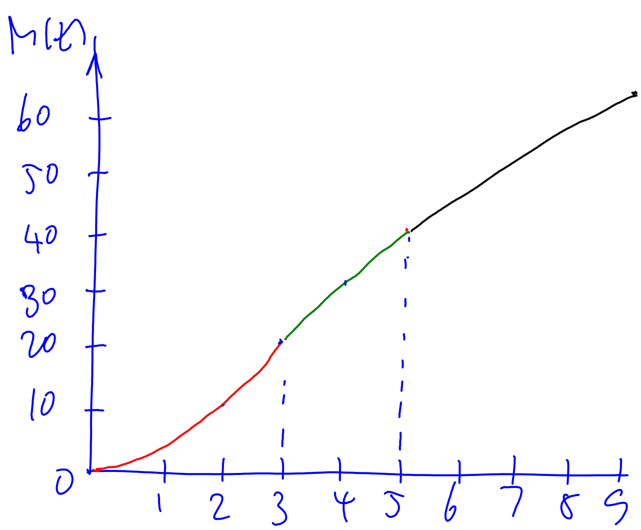
$= Pr\{N(4) - N(2) = 15\} = ?$

$t = 2, \quad s = 2, \quad t+s = 4$

$$Pr\{N(t+s) - N(t) = n\} = e^{-[m(t+s) - m(t)]} \cdot \frac{\{m(t+s) - m(t)\}^n}{n!}$$

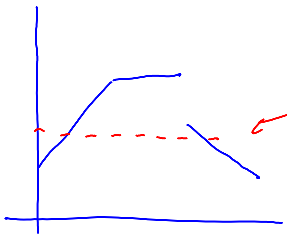
$$m(t) = \int_0^t \lambda(u)\, du$$

$$m(t) = \begin{cases} 4t + t^2, & 0 \le t \le 3, \quad m(3) = 21 \\ 21 + (10t - 30) = 10t - 9, & 3 \le t \le 5, \quad m(5) = 41 \\ 41 + 13t - \frac{1}{2}t^2 - 52.5 = 13t - \frac{1}{2}t^2 - 11.50, & 5 < t \le 9, \quad m(9) = 65 \end{cases}$$



$$m(4) - m(2) = (10 \cdot 4 - 9) - (4 \cdot 2 + 2^2) = 19$$

$$\Rightarrow \Pr\{N(4)-N(2)=15\} = e^{-19}\,\frac{19^{15}}{15!} \simeq 0.065$$



$$\bar{\lambda} = \frac{1}{9}\int_0^9 \lambda(t)\,dt = 7.22$$

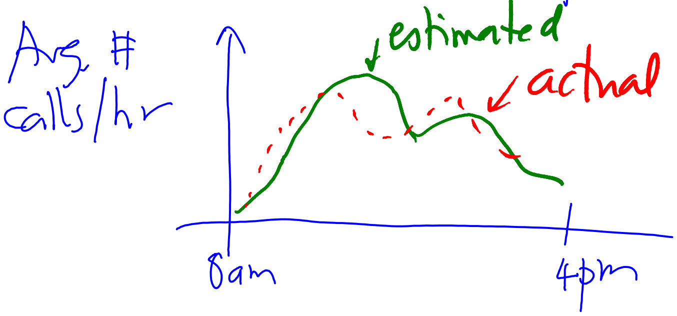
$$\Pr\{N(2)=15\} = e^{-\bar{\lambda}\cdot 2}\,\frac{(\bar{\lambda}\cdot 2)^{15}}{15!} \simeq 0.10$$ larger error!

Ex. Modelling arrivals to a computer system (ECG)
(Kao, SMC-10, #6, 1980) Book p. 61

$M(t)/M/1/4$

arrival $\lambda(t)$ | service | # servers

enough memory to process 1 + hold 3



$$\lambda(t) = 8.924 - 1.584\cos\left(\frac{\pi t}{1.51}\right) + 7.897\sin\left(\frac{\pi t}{3.02}\right) - 10.434\cos\left(\frac{\pi t}{4.53}\right)$$

$$+4.293\cos\left(\frac{\pi t}{6.04}\right)$$

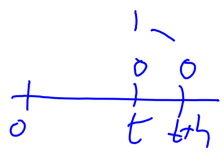
$$\bar{\lambda} = \frac{1}{8}\int_0^8 \lambda(t)\,dt = 10.76/hr \quad : \text{avg. arrival rate}$$

$$\mu = 7 \quad ; \text{service rate}$$

Find $P_n(t) = Pr\{n \overset{ECG}{\text{cust at } t}\}$, $P_0(0)=1$
$P_n(0) = 0,\ n>0$

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda(t)"] & 1 \arrow[l, bend left, "7"] \arrow[r, bend left, "\lambda(t)"] & 2 \arrow[l, bend left, "7"] \arrow[r, bend left, "\lambda(t)"] & 3 \arrow[l, bend left, "7"] \arrow[r, bend left, "\lambda(t)"] & 4 \arrow[l, bend left, "\mu=7"]
\end{tikzcd}$$

μ:



ODE System

$n=0$:
$$P_0(t+h) = P_0(t)\left[1-\lambda(t)h + o(h)\right] + P_1(t)\left[\mu h + o(h)\right]$$

$$\Rightarrow \quad P_0'(t) = -\lambda(t)P_0(t) + \mu P_1(t)$$

$$[P_0'(t), P_1'(t), P_2'(t), \cdots, P_4'(t)] = (P_0(t), P_1(t), P_2(t) \ldots, P_4(t))$$

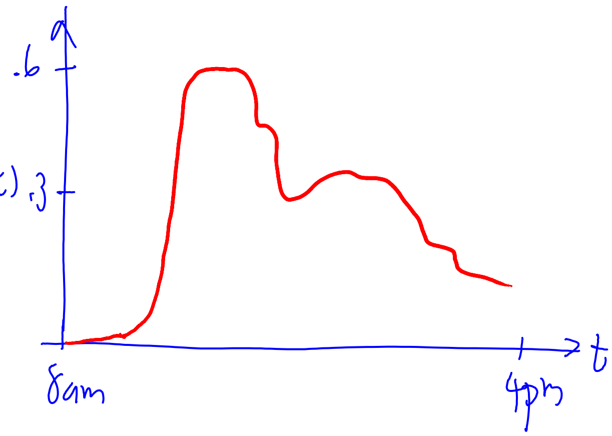
$$\times \begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & -\lambda(t) & \lambda(t) & & & \\
1 & \mu & -[\lambda(t)+\mu] & \lambda(t) & & \\
2 & & \mu & -[\lambda(t)+\mu] & \lambda(t) & \\
3 & & & \mu & -(\lambda(t)+\mu) & \lambda(t) \\
4 & & & & \mu & -\mu
\end{array}$$

$$P'(t) = P(t)\,Q$$
$$[illegible]P(t)$$

Infinitesimal generator matrix

$= Q P(t))$

$P_4(t) = ?$

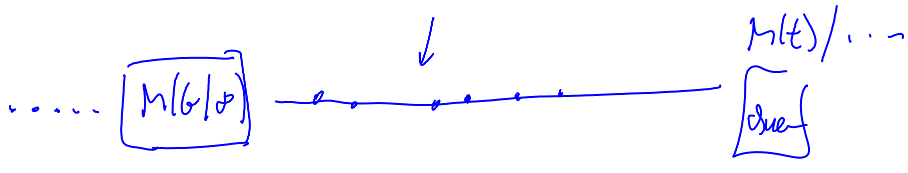


h3 M/G/∞ queue

<u>Proposition</u> Output of an $\overset{\lambda}{M}/G/\infty$ queue is non-homogeneous Poisson with rate

$$\lambda(t) = \lambda \underbrace{G(t)}_{\text{cdf of service time}}$$

<u>Proof</u>. Ross, p.48

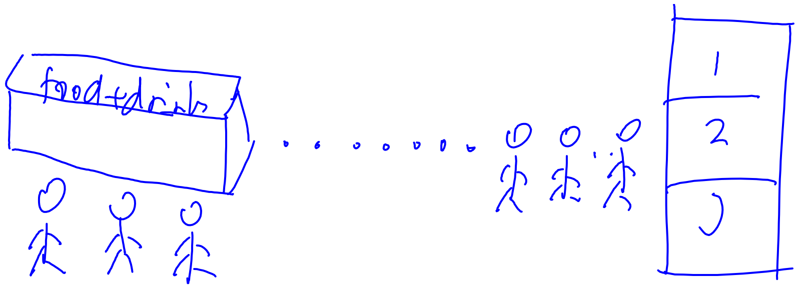


Ex. Small cafetaria in a big university

11:30 – 13:00 opens

Self-service

Cashiers



$M/G/\infty$

$\lambda = 2$ arr/min

$M(t)/M/3$

$\downarrow$

$\lambda(t) = \lambda G(t)$

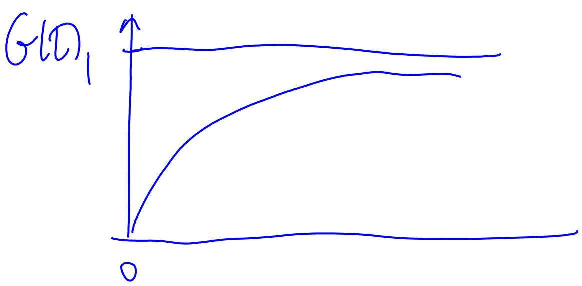
$G(t)$: Erlang$(2, \mu) \sim S_2$

$$E(S_2) = 8 \text{ min} = \frac{2}{\mu} \Rightarrow \mu = \frac{1}{4}$$

$$Var(S_2) = \frac{2}{\mu^2} = 32 \text{ min}^2$$

$$\therefore G(t) = Pr(S_2 \le t) = Pr(N(t) \ge 2)$$

$$= \sum_{k=2}^{\infty} e^{-0.25t} \frac{(0.25t)^k}{k!} = \underbrace{1 - e^{-0.25t}(1 + 0.25t)}$$



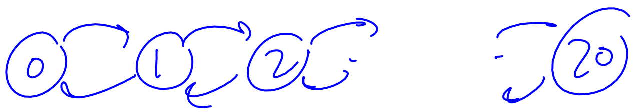
$\therefore$ Arrival rate into $M(t)/M/3$ is

$$\lambda(t) = 2[\quad]$$

$M(t)/M/3/20$

$M(t)/M/3/20$

↑ truncation



Similar to ECG model $\qquad S=3$ (cashier)

$\{P_n(t)\}_{n=0}^{20} \qquad$ 90 minutes $\qquad$ 11:30 — 13:00

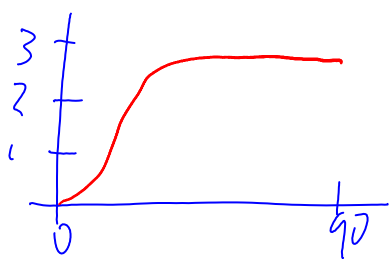
$n=0 \qquad P_0'(t) = -\lambda(t)P_0(t) + \mu P_1(t)$

$1 \le n < S \qquad P_n'(t) = \ \cdots$

$n \ge S \qquad P_n'(t) = \ \cdots \qquad\qquad E(X) = \sum_{x=0}^{\infty} x\,p(x)$

$L(t)$: # cust. in system at t

$$E[L(t)] = \sum_{n=0}^{20} n \cdot P_n(t)$$



e) Compound Poisson process

. Poisson $\{N(t), t \ge 0\}$ $\quad \lambda$

. $\{Y_n\}$ iid with $E(Y)$, $Var(Y)$ (discrete)

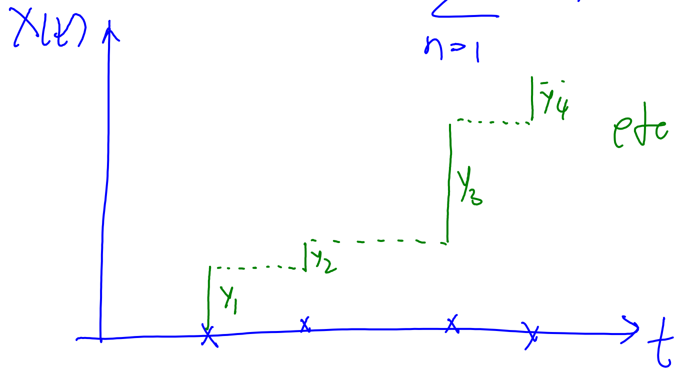
· let $P_Y(z)$ be p.g.f Y,

$$P_Y(z) = E(z^Y) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = Pr(Y=n)$$

· Assume $\{N(t)\}$ & $\{Y_n\}$ are ind't

· Define $\quad X(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}$

$$= \sum_{n=1}^{N(t)} Y_n \quad : \text{Compound Poisson}$$



Find p.g.f. of $X(t)$; i.e., $H_{X(t)}(z) = E\left[z^{X(t)}\right]$

$$H_{X(t)}(z) = E\left[z^{X(t)}\right] = \sum_{n=0}^{\infty} E\left[z^{X(t)} \,\middle|\, N(t)=n\right] Pr\left(N(t)=n\right)$$

$$= \sum_{n=0}^{\infty} E\left[z^{Y_1+\cdots+Y_n}\right] Pr\{N(t)=n\}$$

$$= \sum_{n=0}^{\infty} [\underbrace{E(z^Y)}_{P_Y(z)}]^n \underbrace{Pr\{N(t)=n\}}$$

$$= \sum_{n=0}^{\infty} E\left[z^{Y_1 + \cdots + Y_n} \right] \Pr\{N(t) = n\}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[E\left(z^{Y}\right) \right]^n}_{P_Y(z)} \underbrace{\Pr\{N(t) = n\}}$$

$$= \sum_{n=0}^{\infty} \left(P_Y(z)\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{\left(P_Y(z) \lambda t\right)^n}{n!} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(P_Y(z)\lambda t\right)^n}{n!}$$

$$= e^{-\lambda t} e^{P_Y(z) \lambda t} = e^{\lambda t \left[P_Y(z) - 1\right]}$$

Ex. Stuttering Poisson

$Y_n \rightarrow \Pr(Y = y) = (1-p)\, p^{y-1}, \; y = 1, 2, \ldots$

geometric

$$P_Y(z) = \frac{(1-p)z}{1-pz}, \quad 0 < p < 1$$

$$E(Y) = \frac{1}{1-p}, \quad Var(Y) = \frac{p}{(1-p)^2}$$

Show

$$E[X(t)] = \frac{\lambda t}{1-p}, \quad Var(X(t)) = \frac{\lambda t(1+p)}{(1-p)^2}$$

Exercise:

$$Y_1 \sim \exp(\underset{\text{rate}}{\mu_1}) \; ; \; Y_2 \sim \exp(\mu_2)$$

① $Pr(Y_1 < Y_2) = \frac{\mu_1}{\mu_1+\mu_2}$

② $Pr(\min(Y_1, Y_2) > t) = e^{-(\mu_1+\mu_2)t}$