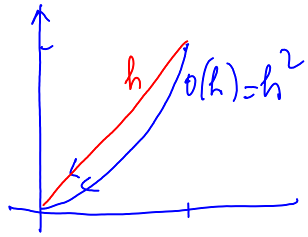
h, $o(h)$?
approaches 0 faster than h does

$o(h) = h^2$



$\frac{h^2}{h} = h$

$\lim_{h \to 0} \frac{o(h)}{h} = 0$

$\Pr\{N(h)=1\} = \lambda h + o(h)$

$\Pr\{N(h) \geq 2\} = o(h)$

Interpretation. Recall $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$\Pr\{N(t)=n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$

$\Pr\{N(h)=1\} = P_1(h) = \lambda h e^{-\lambda h} = \lambda h \left(1 - \lambda h + \frac{1}{2}(\lambda h)^2 + o(h)\right)$
$= \lambda h + o(h)$

$\Pr\{N(h) \geq 2\} = 1 - P_0 - P_1 = o(h)$

$\Pr\{N(h)=1\} = \lambda h + o(h)$ small quant

rate, interval width

Proof of $\Pr\{N(t)=n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n = 0, 1, 2, \ldots$

Induction $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$

$n=1$: $1 = \frac{1}{2}\cdot 1 \cdot 2$ ✓

$n^2 \leq 3n$

n	n^2		$3n$
1	1	<	3
2	4	<	6
3	9	=	9
4	16	>	12

rate' interval width

Proof of $Pr\{N(t)=n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n=0,1,2,\ldots$

$n^2 \le 3n$

n	n^2		$3n$
1	1	$<$	3
2	4	$<$	6
3	9	$=$	9
4	16	$>$	12

Induction $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

$n=1$: $1 = \frac{1}{2}\cdot 1\cdot 2$ ✓

True for $n-1$ $\sum_{i=1}^{n-1} i = \frac{1}{2}(n-1)n$

Show statement is true

$P_n(t) = Pr\{N(t)=n\} = e^{-\lambda t}\left(\frac{(\lambda t)^n}{n!}\right.$

For $n=0$: $P_0(t) = e^{-\lambda t}$?

$$\overset{h}{\overbrace{\;\;}}$$
$0 \quad t \quad t+h$

$P_0(t+h) = Pr\{N(t+h)=0\}$

$= Pr\{N(t)=0, N(t+h)-N(t)=0\}$ ind't incr

$= Pr\{N(t)=0\}\cdot Pr\{N(t+h)-N(t)=0\}$

$= Pr\{N(t)=0\}\cdot Pr\{N(h)=0\}$

$= P_0(t)\left[1-\lambda h+o(h)\right]$

$= P_0(t) - \lambda h P_0(t) + o(h)$

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = f'(t)$$

$$\frac{P_0(t+h)-P_0(t)}{h} = \frac{-\lambda \cancel{h} P_0(t) + o(h)}{\cancel{h}}$$

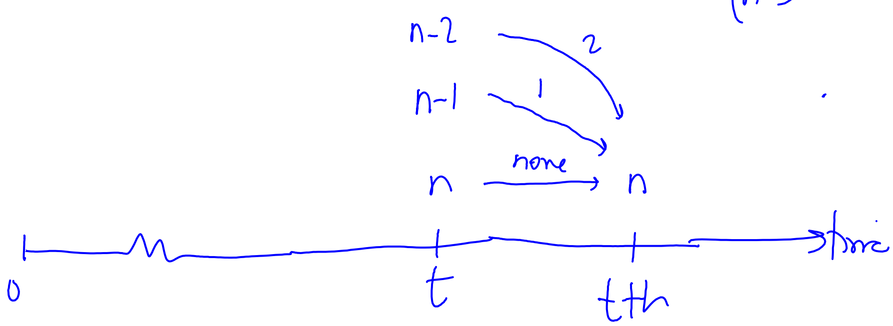
$$\lim \frac{P_0(t+h)-P_0(t)}{} = -\lambda P_0(t) + \lim \cancel{\frac{o(h)}{h}}$$

$h \to 0$ $\qquad h$

$$\left.\begin{aligned} P_0'(t) &= -\lambda P_0(t) \\ P_0(0) &= 1 \end{aligned}\right\} \qquad P_0(t) = e^{-\lambda t}$$

For $n \geq 1$: Assume true for $n-1$

$$\Pr\{N(t) = n-1\} = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$



$$P_n(t+h) = P_n(t) \cdot [1 - \lambda h + o(h)] + P_{n-1}(t) \cdot [\lambda h + o(h)] + P_{n-2}(t) \cdot o(h)$$

$$= P_n(t) - \lambda h P_n(t) + P_{n-1}(t) \lambda h + o(h)$$

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$$

In limit,

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

$$P_n(0) = 0$$

DO IT!

Using $P_{n-1}(t)$ assumption & solving the ODE

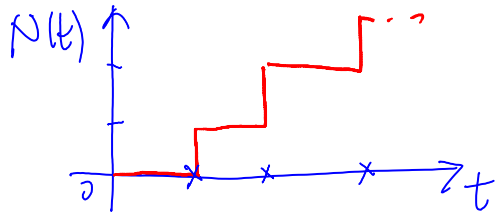
$$P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots$$

$$E[N(t)] = \lambda t.$$

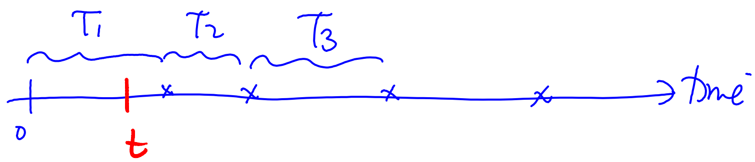
Halmos

tombstone

Q.E.D



(iii) Interarrival times when <u>arrivals Poisson</u>



T_n: time between $(n-1)$st & nth event

$\{T_n ;\ n \geq 1\}$

T_1: $\quad f_{T_1}(t) = ? \longleftarrow \quad F_{T_1}(t) = \Pr\{T_1 \leq t\}$

$\uparrow$

$\bar{F}_{T_1}(t) = \Pr\{T_1 > t\}$ $\swarrow$ find

$$\Pr\{T_1 > t\} = \Pr\{N(t) = 0\} = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$\rightarrow \Pr\{T_1 \leq t\} = 1 - e^{-\lambda t} \rightarrow f_{T_1}(t) = \lambda e^{-\lambda t}$$

Similarly

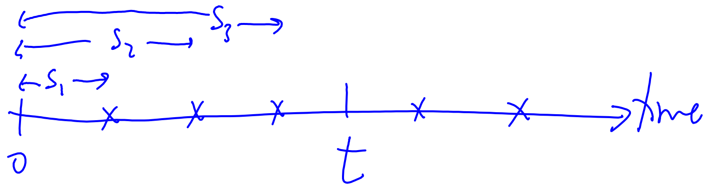
$$\Pr\{T_2 > t \mid T_1 = s\} \overset{\text{SHOW}}{=} e^{-\lambda t}$$

Q: $\quad S_n = T_1 + \cdots + T_n = \sum_{i=1}^{n} T_i$

$$f_{S_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0,$$

iv) An equivalent derivation of $f_{S_n}(t)$:

Find $\Pr\{S_n \le t\} \to$ diff



$$S_3 \le t \iff N(t) \ge 3$$

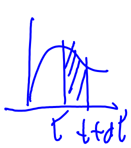
$$S_n \le t \iff N(t) \ge n$$

$$\Pr\{S_n \le t\} = F_{S_n}(t) = \underset{\uparrow \text{Poisson}}{\Pr\{N(t) \ge n\}} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

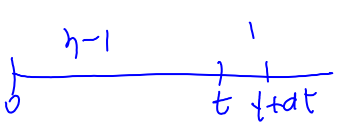
Differentiate $\to$ (Do!)

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad n = 1, 2, \ldots$$

Comment. Re-write as



$$f_{S_n}(t)\,dt = \underbrace{\left\{ e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \right\}} \cdot \lambda\,dt$$

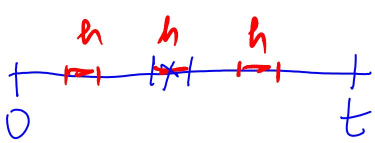


$\boxed{\text{nth}}$ nth

n^{th}

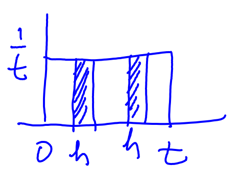
n^{th}

c) Conditional distribution of arrival times



Hint

①



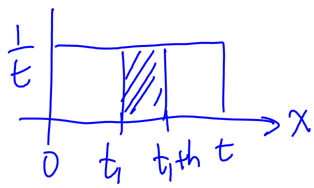
②

X_1: time of occurrence of one

event

Theorem: $f_{X_1|N(t)=1}(x|1) = \frac{1}{t}, \quad 0 \le x \le t$

③ $Pr\{N(h)=1\} \cong \lambda h$



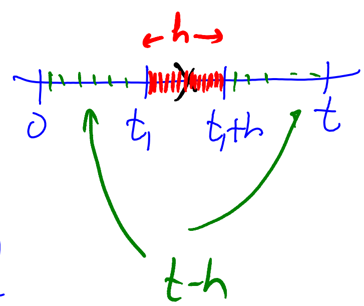
Proof. Recall $\lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = \lim_{h\to 0} \frac{Pr\{t < X \le t+h\}}{h}$

$= f(t)$

$Pr(A|B) = \frac{Pr(A,B)}{Pr(B)}$

Now $Pr\{t_1 \le X_1 \le t_1+h \mid N(t)=1\}$

$$= \frac{Pr\{t_1 \le X_1 \le t_1+h,\ N(t)=1\}}{Pr\{N(t)=1\}}$$



$$= \frac{Pr\{1 \text{ in } [t_1, t_1+h), \text{ zero elsewhere}\}}{Pr\{N(t)=1\}}$$

$$= \frac{\{\lambda h e^{-\lambda h}\}\{e^{-\lambda(t-h)}\}}{\lambda t e^{-\lambda t}} = \frac{h}{t}$$

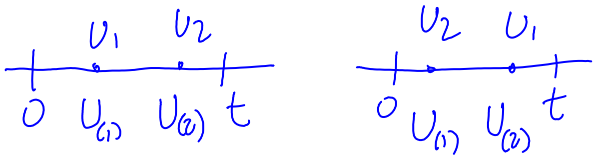
$$\therefore \lim_{h\to 0} \frac{Pr\{\ \cdot \mid \cdot\ \}}{h} = f_{X_1|N(t)=1}(x|1) = \frac{\frac{h}{t}}{h} = \frac{1}{t}, \quad 0<x<t$$

QED

Generalization: Preliminary results

Let U_1, U_2 be uniform over $(0,t)$

$U_{(1)}, U_{(2)}$ " order statistics for U_1 and U_2



$f_{U_1}(u_1) = \frac{1}{t}$

$f_{U_2}(u_2) = \frac{1}{t}$

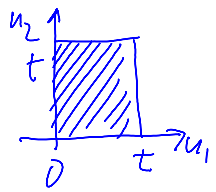
$U_{(1)} = \min(U_1, U_2)$

$U_{(2)} = \max(U_1, U_2)$

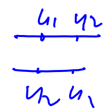
$t = 10,$ $U_1 = 7$ $U_2 = 3$

$U_{(1)} = 3, \; U_{(2)} = 7$

Fact $\quad f_{U_1,U_2}(u_1,u_2) = \frac{1}{t} \cdot \frac{1}{t} = \frac{1}{t^2}, \quad \begin{array}{l} 0 \le u_1 \le t \\ 0 \le u_2 \le t \end{array}$



Show $\displaystyle\int_0^t \int_0^t f_{U_1,U_2}(u_1,u_2)\,du_1\,du_2 = 1$

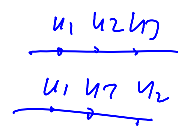


In general $\quad f_{U_1,\ldots,U_n}(u_1,\ldots,u_n) = \frac{1}{t^n}, \quad 0 \le u_i \le t, \; i=1,\ldots,n$

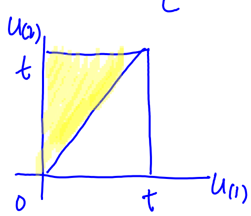
Theorem For order statistics $U_{(1)}$ and $U_{(2)}$,

$$f_{U_{(1)},U_{(2)}}(u_{(1)},u_{(2)}) = \frac{2!}{t^2}, \quad 0 \le u_{(1)} \le u_{(2)} \le t$$

U_1, U_2, U_3



$\vdots$ $\quad 3!$



Show $\displaystyle\int_0^t \int_0^{u_{(2)}} \frac{2!}{t^2}\,du_{(1)}\,du_{(2)} = 1$

Proof. Skip ∎

In general $\quad f_{U_{(1)},\ldots,U_{(n)}}(u_{(1)},\ldots,u_{(n)}) = \frac{n!}{t^n}, \; 0 \le u_{(1)} \le \cdots \le u_{(n)} \le t$

Remark

$$\boxed{\sum_{i=1}^{n} U_{(i)} = \sum_{i=1}^{n} U_i \Rightarrow E\left(\sum_{i=1}^{n} U_{(i)}\right) = E\left(\sum_{i=1}^{n} U_i\right)}$$

Theorem Given that $N(t)=n$, the n ordered arrival times $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics corresponding to the n i.i.d. uniform r.v.'s over $(0,t)$, i.e.,

$$f_{S_1,\ldots,S_n|N(t)=n}(s_1,\ldots,s_n|n) = \frac{n!}{t^n}, \quad 0 \le s_1 \le \cdots \le s_n \le t$$

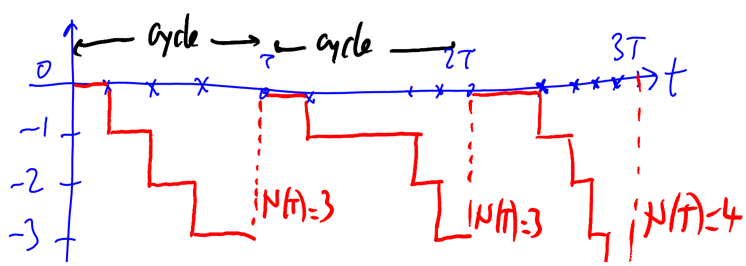
Ex. Periodic-review inventory policy

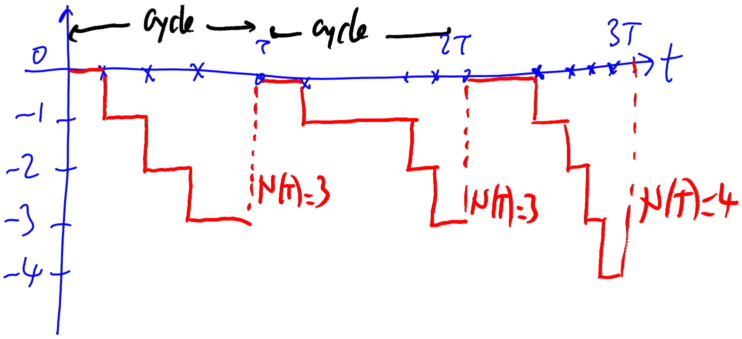
Review inventory at $T, 2T, 3T, \ldots$ (T: review length)

Suppose all demand is backordered (backlogged) & order is placed at end of period to satisfy all demands.

Demand Poisson with rate λ

Unit purchase cost $= c$





Renewal reward Thm (RRT)

$$AC(T) = C(T) = \frac{E(\text{cycle cost})}{E(\text{cycle length})}$$

- Cycle length $= T$ (time)

λ : $\left(\frac{\text{units}}{\text{time}}\right)$

c : $\left(\frac{\$}{\text{unit}}\right)$ $\qquad E[N(T)] = \lambda T$

- Cycle cost

(i) Order cost : K [\$]

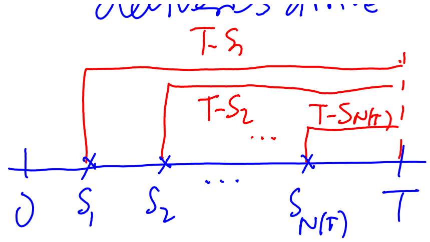
(ii) Purchase cost : $cN(T) \Rightarrow E(\text{Purchase cost}) = c\lambda T$

$$\frac{\$}{\cancel{u}} \cdot \frac{\cancel{u}}{\cancel{t}} \cdot \cancel{t}$$

(iii) Backorder cost

$\hat{\pi}$: cost of backorder/unit/time $\quad \frac{\left[\frac{\$}{u}\right]}{(t)} = \left[\frac{\$}{u} \cdot \frac{1}{t}\right]$

$W(T)$: total waiting time of all customers until deliveries arrive



$$W(T) = (T-S_1) + (T-S_2) + \cdots + (T-S_{N(T)})$$
$$= \sum_{i=1}^{N(T)} (T-S_i)$$