2013-09-16

Monday, September 16, 2013
12:00 PM

d) Moment generating functions (m.g.f.)

$$\phi(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\,dx \qquad X \geq 0$$

$$\phi'(t) = \int_0^\infty x e^{tx} f(x)\,dx \qquad \text{(Leibniz's rule)}$$

$$= E(X e^{tX})$$

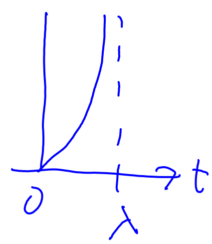
$$\phi'(0) = E(X)$$

$$\phi''(t) = \int_0^\infty x^2 e^{tx} f(x)\,dx = E(X^2 e^{tX})$$

$$\phi''(0) = E(X^2)$$

$$Var(X) = \phi''(0) - (\phi'(0))^2$$

Ex. $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $x \geq 0$



$$\phi(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

$$\phi'(t) = \frac{\lambda}{(\lambda-t)^2}, \quad \phi'(0) = \frac{1}{\lambda} = E(X)$$

$$\phi''(t) = \frac{2\lambda}{(\lambda-t)^3}, \quad \phi''(0) = \frac{2}{\lambda^2} = E(X^2)$$

$$Var(X) = \frac{1}{\lambda^2}, \qquad SD(X) = \frac{1}{\lambda}$$

e) Probab. generating function (p.g.f.)

Consider a discrete r.v and its p.d.f. $\{p_k\}_{k=0}^{\infty}$
↓ nonnegative

0	1	2	3	...
p_0	p_1	p_2	p_3	...

Ex. $\Pi(z) = \dfrac{4}{(2-z)(3-z)^2}$

$\Pi(1) = \dfrac{4}{1 \cdot 2^2} = 1$

$$\Pi(z) = \sum_{k=0}^{\infty} z^k p_k = E(z^X)$$

$$= p_0 + z p_1 + z^2 p_2 + z^3 p_3 + \cdots$$

$$\Pi(1) = \sum_{k=0}^{\infty} p_k = 1$$

$$\Pi'(z) = \sum_{k=0}^{\infty} k z^{k-1} p_k, \quad \Pi'(1) = \sum_{k=0}^{\infty} k p_k = E(X)$$

$$\Pi''(z) = \sum_{k=0} k(k-1) z^{k-2} p_k \Rightarrow \Pi''(1) = E(X^2) - E(X)$$

show

$$Var(X) = \Pi''(1) + \Pi'(1) - [\Pi'(1)]^2$$

Ex. Non-probab $\quad \Pi(z) = \sum_{k=0}^{\infty} z^k f_k$

$f_n = 1, \quad n = 0, 1, 2, \ldots$

$$\Pi(z) = \sum_{0}^{\infty} z^k = \frac{1}{1-z}, \quad |z| < 1$$

Ex. $f_n = \alpha^n, \quad n = 0, 1, 2, \ldots$

$$\Pi(z) = \sum_{k=0}^{\infty} z^k \alpha^k = \sum_{k=0}^{\infty} (\underbrace{\alpha z}_{y})^k = \frac{1}{1 - \alpha z}$$

Ex. If $\Pi(z) = \frac{1}{1+\frac{1}{4}z} \Rightarrow f_n = \left(-\frac{1}{4}\right)^n$

Ex. Probabilistic ex

$$\Pi(z) = \frac{4}{(2-z)(3-z)^2}, \quad \Pi(1) = 1$$

$\downarrow$

partial fractions

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/PGF.mw

$\Downarrow$

$$p_n = 2\left(\frac{1}{2}\right)^n - \left(\frac{4}{3}\right)\left(\frac{1}{3}\right)^n - \left(\frac{4}{9}\right)\left(\frac{1}{3}\right)^n(n+1), \quad n = 0, 1, 2, \ldots$$

$$\sum_0^\infty p_n = 1$$

$$\Pi'(z) = \frac{4(3z-7)}{(z-2)^2(z-3)^3}, \quad \Pi'(1) = E(X) = 2$$

$$\Pi''(1) = \frac{11}{2}, \quad Var(X) = \frac{7}{2}$$

Picking the prob's

$$\Pi(z) = p_0 + z p_1 + z^2 p_2 + z^3 p_3 + z^4 p_4 + \cdots$$

$$\Pi(0) = p_0$$

$$\Pi'(z) = p_1 + 2z p_2 + 3z^2 p_3 + \cdots$$

$$\Pi'(0) = p_1$$

$$\Pi''(z) = 2p_2 + 6z p_3 + \cdots$$

$$\Pi''(0) = 2p_2$$

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$$\Rightarrow \quad p_k = \frac{1}{k!}\,\frac{d^k\,\Pi(z)}{dz^k}\Bigg|_{z=0}, \quad k=0,1,2,\ldots$$

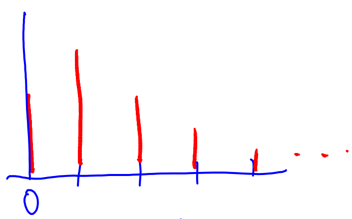
$$p_0 = \Pi(0) = \frac{2}{9}$$

$$p_1 = \frac{d\Pi}{dz}\Bigg|_{z=0} = \frac{7}{27}$$

$$p_2 = \frac{1}{2}\,\frac{d^2\Pi}{dz^2}\Bigg|_{z=0} = \frac{11}{54}, \text{ etc}$$

Ex. Poisson

$$\Pr(X=k) = p_k = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k=0,1,2,\ldots$$



$$\Pi(z) = e^{-\lambda}e^{z\lambda} \qquad \text{Show}$$

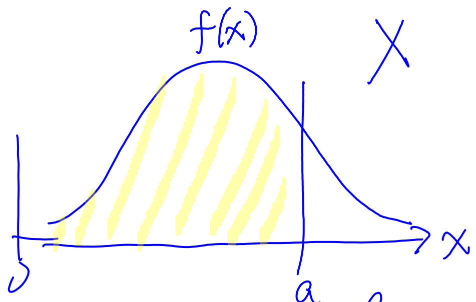
$$\Pi'(z) = e^{-\lambda}\lambda e^{z\lambda}, \qquad \Pi'(1) = \lambda$$

$$\Pi''(z) = e^{-\lambda}\lambda^2 e^{z\lambda}, \qquad \Pi''(1) = \lambda^2 = E(X^2) - E(X)$$

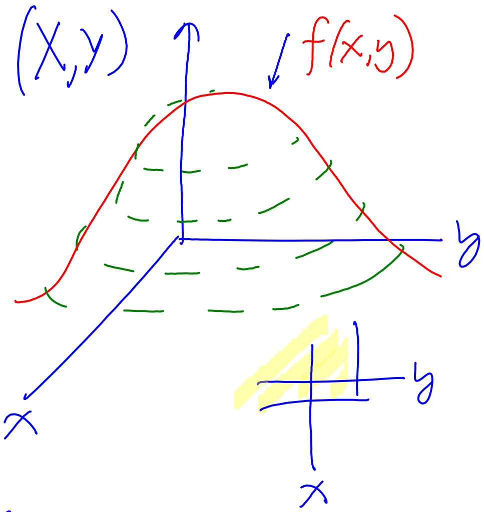
$$\Rightarrow E(X^2) = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \lambda$$

f) Joint random variables



$$F(a) = \Pr(X \le a) = \int_0^a f(x)\,dx$$



$$F(a,b) = \Pr(X \le a,\ Y \le b)$$

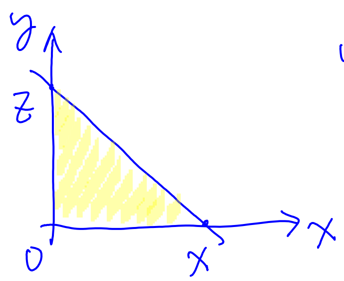
Ex. X, Y ind't

$$\underset{f(x)}{X},\ \underset{g(y)}{Y}$$

$$Z = X + Y$$

Find $H_Z(z)$ & $h_Z(z)$

$$H_Z(z) = H_{X+Y}(z) = \Pr\{X + Y \le z\} = \iint_{x+y \le z} f(x)\,g(y)\,dy\,dx$$



$y \le z - x$

show

$$h_Z(z) = H_Z'(z) = \int_0^z g(z-x) f(x)\,dx$$

convolution integral

$$= \int_0^z f(z-y)\,g(y)\,dy$$

http://profs.degroote.mcmaster.ca/ads/parlar/courses/Q771/ChapterComments/SumOfRVs.mw

q) Laplace transforms

X is continuous with $f(x)$, $x \geq 0$

$$\tilde{f}(s) = \int_0^\infty e^{-sx} f(x)\,dx = E(e^{-sX})$$

$$\tilde{f}(0) = \int_0^\infty f(x)\,dx = 1$$

$$\tilde{f}'(s) = \int_0^\infty -x e^{-sx} f(x)\,dx$$

$$\tilde{f}'(0) = -\int_0^\infty x f(x)\,dx = -E(X)$$

$$\tilde{f}''(0) = E(X^2)$$

Ex. $f(x) = \lambda e^{-\lambda x}$

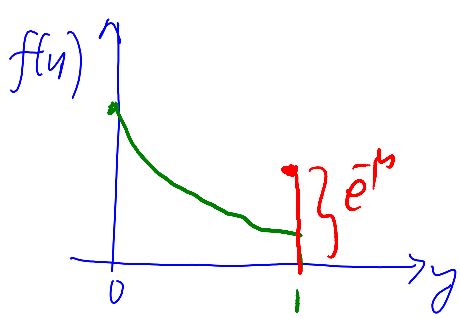
$$\tilde{f}(s) = \frac{\lambda}{\lambda + s}, \quad \tilde{f}'(0) = -\frac{1}{\lambda} = -E(X)$$

$$\tilde{f}''(0) = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{1}{\lambda^2}$$

Ex. Car battery

$$f(y) = \begin{cases} \mu e^{-\mu y}, & 0 \leq y < 1 \\ e^{-\mu} & y = 1 \end{cases}$$



$$\tilde{f}(s) = \int_0^\infty e^{-sy} f(y)\,dy = \int_0^1 e^{-sy} \mu e^{-\mu y}\,dy + e^{-s \cdot 1} \cdot e^{-\mu}$$

$$= \frac{\mu + e^{-(s+\mu)} \cdot s}{s+\mu}$$

$$\therefore \quad \tilde{f}'(0) = -\frac{1-e^{-\mu}}{\mu} = -E(Y)$$

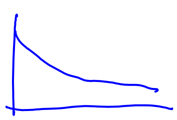
$$\Rightarrow \quad E(Y) = \frac{1-e^{-\mu}}{\mu} < \frac{1}{\mu}$$

Some Tauberian theorems

a) Final value thm

When $\lim_{t\to\infty} f(t)$ exists $\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\tilde{f}(s)$

Ex. $f(t) = \lambda e^{-\lambda t}$, $\tilde{f}(s) = \dfrac{\lambda}{\lambda+s}$

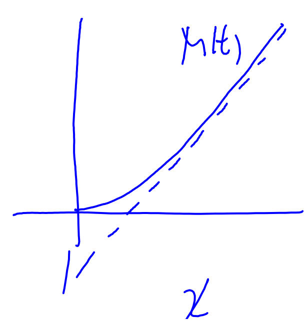


$\lim_{t\to\infty} f(t) = 0$ $\qquad$ $\lim_{s\to 0} s \dfrac{\lambda}{\lambda+s} = 0$

b) Asymptotic rate thm

When limit exists, $\lim_{t\to\infty} \dfrac{f(t)}{t} = \lim_{s\to 0} s^2 \tilde{f}(s)$

Ex. $M(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$, $t \geq 0$



(i) $\lim_{t\to\infty} M(t)$ doesn't exist!

(i) $\lim_{t\to\infty} M(t)$ doesn't exist!

(ii) $\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{s\to 0} s^2 \tilde{M}(s) = \lim_{s\to 0} \frac{\cancel{s^2}}{\cancel{s^2}(s+2)} = \frac{1}{2}$

$$\tilde{M}(s) = \int_0^\infty e^{-st} M(t)\,dt = \boxed{\frac{1}{s^2(s+2)}}$$

Exercise: $Z = X + Y$ indep

$$h_Z(z) = \int_0^z g(z-x) f(x)\,dx$$

$$\tilde{h}(s) = \tilde{f}(s)\,\tilde{g}(s)$$ Convolution prop

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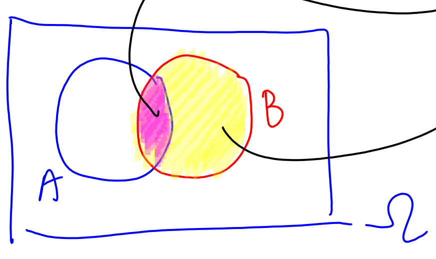
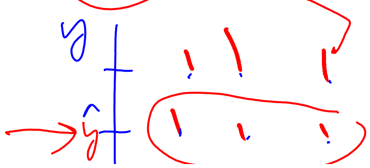
4) Conditional Prob & expectation

A, B events $\quad \Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$ if $\Pr(B) > 0$

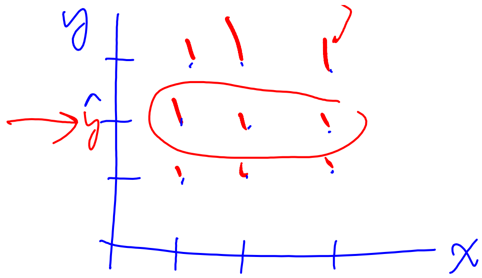


1. Discrete r.v.'s

$X, Y \qquad p(x,y) = \Pr\{X = x, Y = y\}$



$$\sum\sum p(x,y) = 1$$

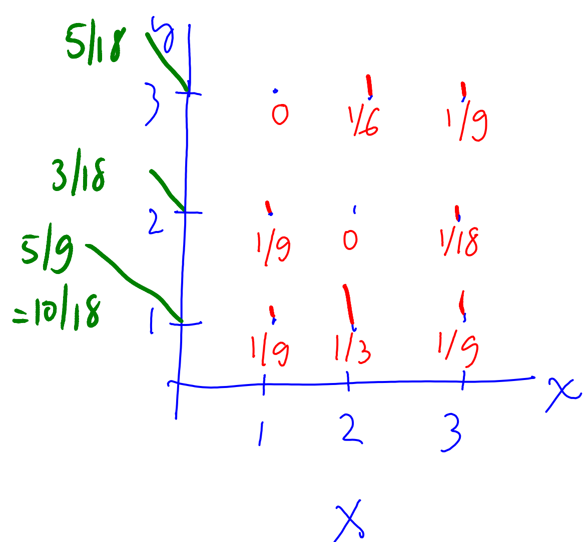


$$\sum\sum p(x,y)=1$$

$$p_{X|Y}(x|\hat{y}) = \Pr\{X=x \mid Y=\hat{y}\} = \frac{p(x,\hat{y})}{p_Y(\hat{y})} \leftarrow \text{marginal}$$

$$p_Y(\hat{y}) = \sum_x p(x,\hat{y})$$

Ex.

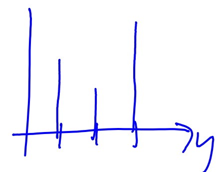


Marginal

$p_Y(3) = 5/18$

$p_Y(2) = 3/18$

$p_Y(1) = 5/9$



$$p_{X|Y}(1|1) = \frac{1/9}{5/9} = \frac{1}{5}, \quad p_{X|Y}(2|1) = \frac{3}{5}, \quad p_{X|Y}(3|1) = \frac{1}{5}$$

$$E(X|Y=1) = 2$$

Ex. ind't $\begin{cases} X & \text{Poisson } \lambda_1 \quad \text{female } ♀ \\ Y & \text{〃} \quad \lambda_2 \quad \text{male } ♂ \end{cases}$

$$E(X \mid X+Y=n) = ?$$

$$\left(\lambda_1\right)^{x} \left(\lambda_2\right)^{n-x}$$

$$\Pr\{X=x \mid X+Y=n\} = \binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$ Show

$$\mathrm{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

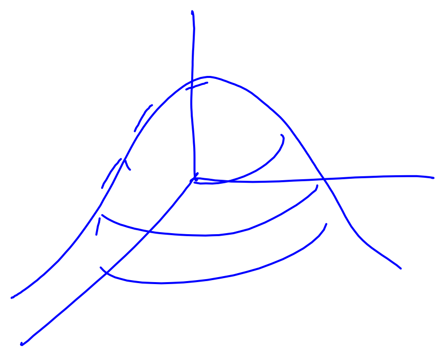
$$E(X \mid X+Y=n) = n\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$n=20$, $\lambda_1=5$, $\lambda_2=10$

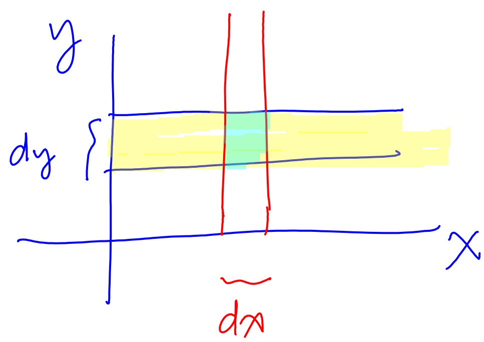
$E(\cdot|\cdot) = 6.67$

2. Cont

X, Y joint $f(x,y)$



$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$



$f_{X|Y}(x|y)\,dx\,dy$

$$= \frac{f(x,y)}{f_Y(y)}\,dx\,dy$$

$$\Rightarrow f_{X|Y}(x|y)\,dx = \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

Ex. $f(x,y) = \dfrac{e^{-x/y} \cdot e^{-y}}{y}$, $0 < x < \infty$, $0 < y < \infty$

$\int_0^\infty \int_0^\infty f(x,y)\,dy\,dx = 1$ Show

Find $E(X|Y=y) = \int_0^\infty x \underbrace{f_{X|Y}}(x|y)\,dx$

$$\frac{f(x,y)}{f_Y(y)} \longrightarrow \int_0^\infty f(x,y)\,dx = e^{-y}$$

$E(X|Y=y) = y$

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Next time: $E(X) = E_Y\left(E_{X|Y}(X|Y)\right)$