ΣΤΩΚΩσ (target)

0. What is a stochastic process (SP)

A SP is an infinite family of r.v.'s indexed by a time parameter

$\{X_n ; n=0,1,2,\ldots\}$ Discrete-time

$\{X(t) ; t \geq 0\}$ Cont. "

Time \ State	D	C
D	DTMC	
C	CTMC	BM

Ex.1. Random Walk (Coin toss)

$$Z_n = \begin{cases} +1 & \text{Coin H} \\ -1 & \text{Coin T} \end{cases}$$

X_n: wealth after nth toss

$X_0 = 0$

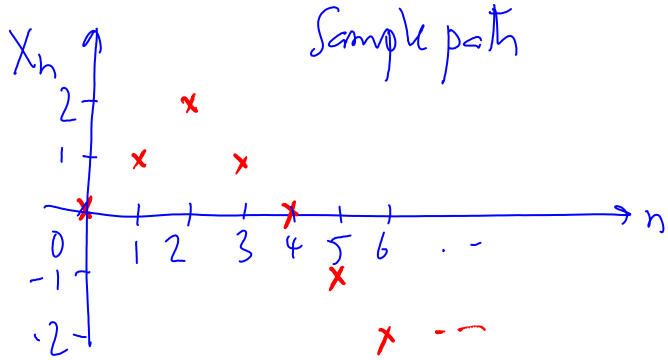
$X_1 = X_0 + Z_1$

$X_2 = X_1 + Z_2$

$\vdots$

$X_n = X_{n-1} + Z_n$

Sample path



D.T.

D.SS.

Ex.2 Insurance Risk

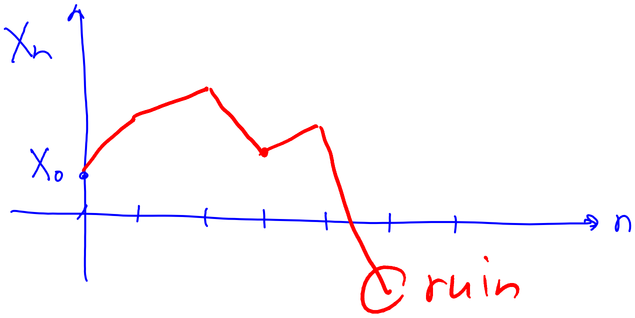
X_0: initial capital (observe system at end of month)

I_n: income during n

C_n: claims " "

$$X_n = X_0 + (I_1 - C_1) + (I_2 - C_2) + \cdots + (I_n - C_n)$$

$$= X_0 + \sum_{1}^{n} (I_i - C_i)$$

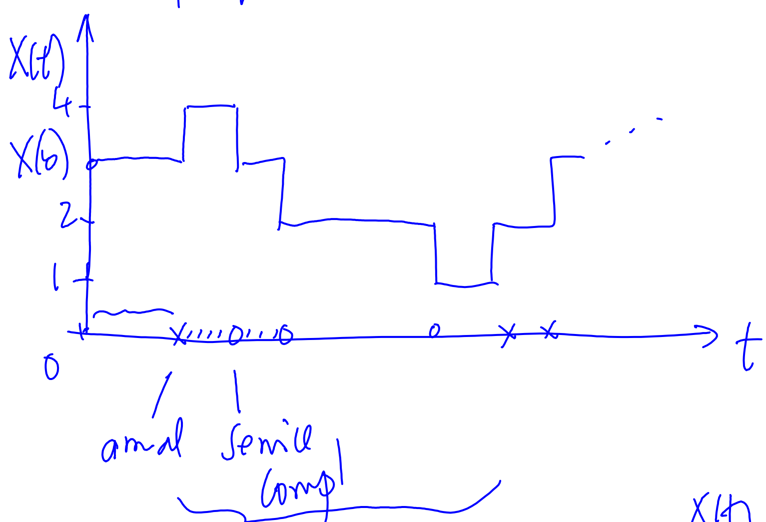


D.T.

C.SS

EX.3 Queueing system

$X(t)$: # people in system at t



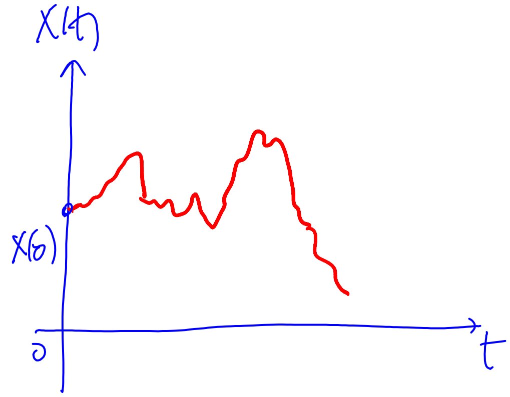
Decision

CT

D.SS

EX.4. Dam problem

$X(t)$: am't of water at t



C.T

C.SS

1. Review of Prob. Theory

a) Sample spaces & Prob. measures

· Sample space Ω

Ex. $\Omega = \{H, T\}$

outcomes

· Event (subset of Ω)

Ex. Die $\Omega = \{1, 2, \cdots, 6\}$

Events have a natural structure

· Ω is event (sure thing)

· $A_1 = \{2, 4, 6\}$, $A_1^C = \{1, 3, 5\}$ complement

· $A_1 = \{1, 2\}$, $A_2 = \{5, 6\}$, $A_1 \cup A_2 = \{1, 2, 5, 6\}$ event

Consider a family of events $\mathcal{F}$ such that

$$\left.\begin{array}{l} \Omega \in \mathcal{F} \\ A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F} \quad (A^C = \Omega \setminus A) \\ \text{if } A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \end{array}\right\} \sigma\text{-field}$$

(ϕ: impossible)

Ex. $\Omega = \{a, b, c\}$

Ω

$\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$: Power set

Ex. $\{\phi, \{a\}, \{b\}, \{a,b\}\}$?

$\{a,b\}^c = \{c\}$ not σ-field

Ex. $\{\phi, \{a,b\}, \{c\}, \{a,b,c\}\}$ ✓

Def. Given Ω and $\mathcal{F}$, a prob measure is a function $\Pr\{\ \}$ which assigns a # for each event $A \in \mathcal{F}$

(a) $0 \le \Pr\{A\} \le 1$

(b) $\Pr\{\Omega\} = 1$

(c) For $A_1, A_2, \ldots,$ $A_i \cap A_j = \phi$ for $i \ne j$ (disjoint)

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum \Pr\{A_i\}$$

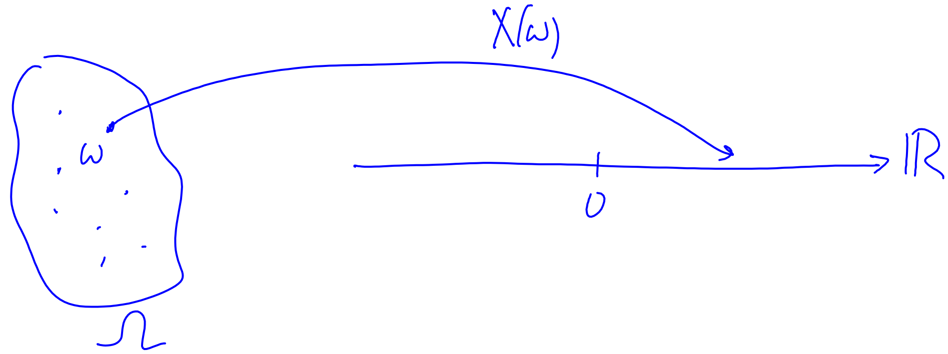
(If $A \cap B \ne \phi$, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$)

$(\Omega, \mathcal{F}, P)$: prob. space

(b) Random Variable + Distribution functions

Random Variable $X : \Omega \to \mathbb{R}$ $\qquad\qquad X(\omega)$

function

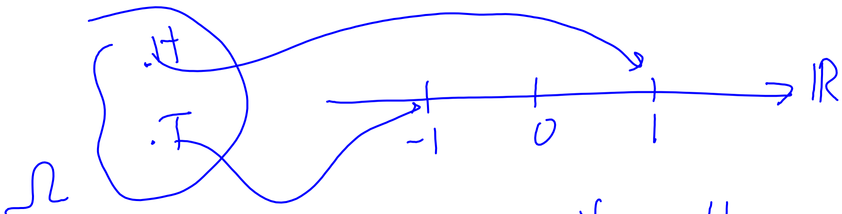


$$\Pr(X \le x)$$

Ex. Coin

H.  : \$1

T:  : \$-1

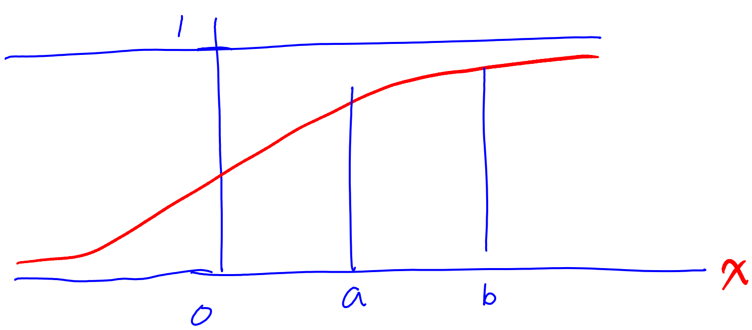


$$X(\omega) = \begin{cases} +1 & \text{if } \omega = H \\ -1 & \text{if } \omega = T \end{cases}$$

$$X(\omega) \to X$$

X : $F(\cdot)$ distribution funct (c.d.f.)

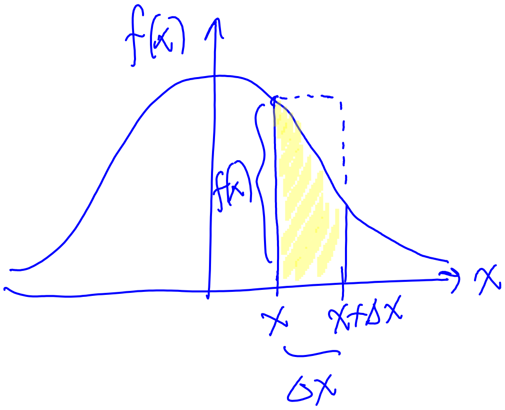
$$F(x) = \Pr(X \le x), \quad -\infty < x < \infty$$



$$\Pr(a < X \le b) = \Pr(X \le b) - \Pr(X \le a)$$
$$= F(b) - F(a)$$

When X is cdf F is continuous + differentiable, F has derivative (prob. dens. func) f

$$f(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = F'(x)$$



$$f(x)\Delta x = F(x+\Delta x) - F(x) + o(\cancel{\Delta x})$$

$$\boxed{\begin{aligned} f(x)\Delta x &\simeq F(x+\Delta x) - F(x) \\ &= Pr(x \le X \le x+\Delta x) \end{aligned}}$$

$$Pr(x \le X \le x+\Delta x) = \int_x^{x+\Delta x} f(u)\,du$$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

c) Moments

$$E(X) = \int_0^\infty x f(x)\,dx \quad : \text{mean}$$

$$E(X^n) = \int_0^\infty x^n f(x)\,dx$$

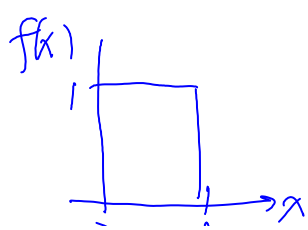
$$\int_0^\infty (x - E(x))^2 f(x)\,dx \quad : \text{variance} :$$

$$Var(X) = E(X^2) - [E(X)]^2$$

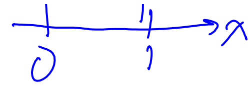
Ex. $f_T(t) = \lambda e^{-\lambda t}, \; t \ge 0, \; \lambda > 0$ $\qquad \int_0^\infty \lambda e^{-\lambda t}\,dt =$

Show $E(T) = \dfrac{1}{\lambda}$

Ex. X uniform over $[0,1]$

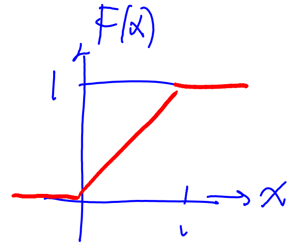


Ex. X uniform on [0,1]



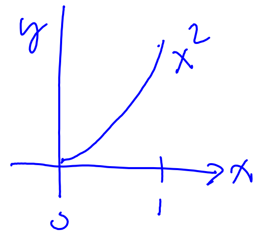
$f(x) = 1, \quad 0 \le x \le 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ x, & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$



Find $E(X^2)$ $\qquad E(X) = \frac{1}{2}$

$Y = X^2$



$F(x) = \Pr(X \le x)$

Find $f_Y(y)$, $E(Y) = \int y f_Y(y)\,dy$

cdf $F_Y(a) = \Pr(Y \le a) = \Pr(X^2 \le a)$

$= \Pr(X \le \sqrt{a}) = \sqrt{a}$

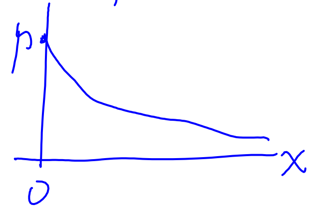
pdf $f_Y(a) = F'(a) = \frac{1}{2} a^{-\frac{1}{2}} = \frac{1}{2\sqrt{a}}, \quad 0 < a \le 1$

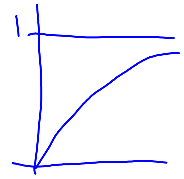
$$E(Y) = \int_0^1 \frac{1}{2\sqrt{y}} \cdot y\,dy = \frac{1}{3}$$

(Easier $E(X^2) = \int_0^1 x^2 \cdot 1\,dx = \frac{1}{3}$

Ex. X : lifetime of a car battery $\sim$ expon(μ)

$f_X(x) = \mu e^{-\mu x}, \quad x \ge 0$



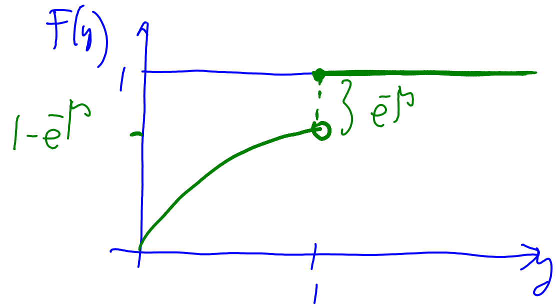


$E(X) = \frac{1}{\mu}$

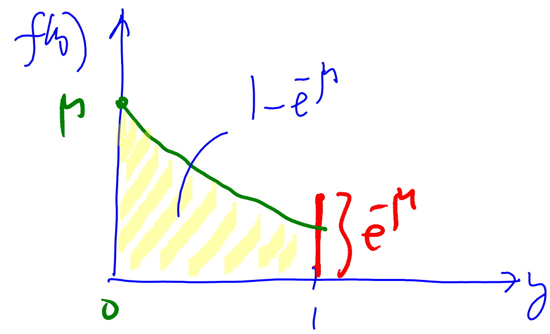
$E(X) = \frac{1}{\mu}$

Y: actual lifetime; $\quad Y = \min(X, 1)$

$$F_Y(y) = \begin{cases} 1 - e^{-\mu y}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$



$$f_Y(y) = \begin{cases} \mu e^{-\mu y}, & 0 \le y < 1 \\ e^{-\mu}, & y = 1 \end{cases}$$



Show $E(Y) = \frac{1}{\mu}\left(1 - e^{-\mu}\right) < \frac{1}{\mu}$

$$\int_0^1 \mu e^{-\mu y}\, dy = 1 - e^{-\mu}$$

$\Pr(X \le y) = 1 - e^{-\mu y}$, $\quad \Pr(X \le 1) = 1 - e^{-\mu}$

$\Pr(X > 1) = e^{-\mu}$