2012-06-29

Friday, June 29, 2012
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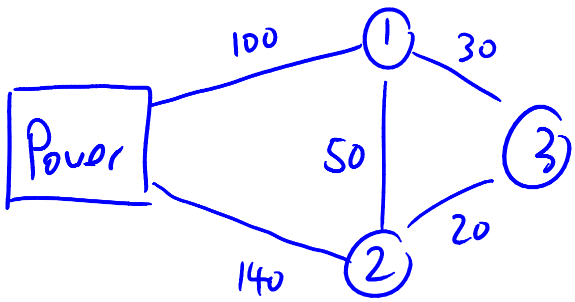
IV. Cooperative Games with transferable utility

- How to divide savings?

Ex. Three cities



S	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$c(S)$	100	140	130	150	130	150	150
$v(S)$ savings	0	0	0	90	100	120	220

How to allocate savings $v(12)$

Consider $S=\{1,2\}$, $v(\{1,2\})=v(S)=90$

savings if they cooperate

$x_1+x_2 \geq 90$

Def. A coop. game with transferable utility

Def A coop.-game with transferable utility (TU) is denoted by (N, v), where

$N = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$ (players),

v: characteristic function

$v(S)$: worth of coalition S

Coalition N: grand coalition

x_i: payoff to $i \in S$

a) The Core

Ex. Three cities

Suppose #3 proposes, $x_1 = 40$

$x_2 = 40$ $\quad v(N) = 220$

$x_3 = 140$

1 & 2 protest, since $v(1,2) = 90 > 80 = x_1 + x_2$

$\therefore$ So, $(40, 40, 140)$ is <u>not</u> in core

Core:

$$C = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \geq 0,$$

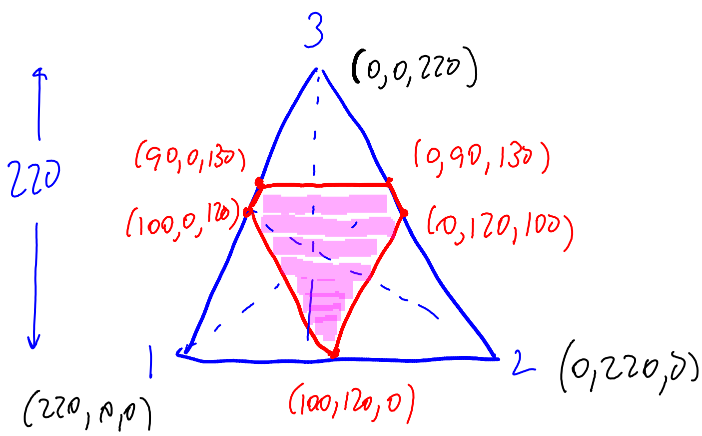
$x_1+x_2 \geq 90, \quad x_1+x_3 \geq 100, \quad x_2+x_3 \geq 120,$

$x_1+x_2+x_3 = 220\}$

$$x_1+x_2 \geq 90 \iff x_3 \leq 220-90=130$$

$$x_1+x_3 \geq 100 \iff x_2 \leq 220-100=120$$

$$x_2+x_3 \geq 120 \iff x_1 \leq 220-120=100$$



Consider $S \subseteq N = \{1, 2, \ldots, n\}$ and

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$\boxed{x(S) := \sum_{i \in S} x_i}$$

Def For a TU-game (N, v), the payoff vector $x \in \mathbb{R}^n$ is

imputation $\{$ • efficient if $x(N) = v(N)$

imputation $\begin{cases} \bullet\ \underline{\text{efficient}} & \text{if } x(N)=v(N) \\ \bullet\ \underline{\text{individually rational}} & \text{if } x_i \geq v(\{i\}) \end{cases}$

$\bullet$ $\underline{\text{coalitionally}}$ " if $x(S)\geq v(S)$ $\forall S\neq\emptyset$

The core of (N,v) is the set

$$C(N,v)=\{x\in\mathbb{R}^n;\ x(N)=v(N),$$
$$x(S)\geq v(S)$$
$$\text{for all } S\subseteq N, \text{ such that } S\neq\emptyset\}$$

Ex. Core empty (divide €, $, ¥, TL)

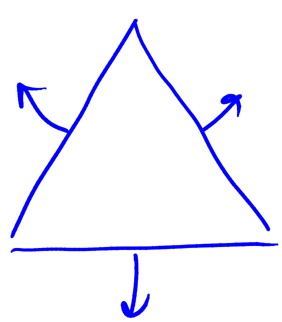
$$v(1)=v(2)=v(3)=0$$
$$v(12)=v(13)=v(23)=1$$
$$v(123)=1$$

No core!

$x_1+x_2\geq 1 \iff x_3\leq 0$

$x_1+x_3\geq 1 \qquad x_2\leq 0$

$x_2+x_3\geq 1 \qquad x_1\leq 0$



(Final, Q3: $\theta = \frac{1}{2} = 1-\theta$)

(b) Shapley Value

Idea: "Avg. marginal contribution"

Ex. Three cities

S	ϕ	1	2	3	1,2	1,3	2,3	1,2,3
$v(S)$	0	0	0	0	90	100	120	220

P1 $\quad v(1) - v(\phi) = 0 - 0 = 0$

P2 $\quad v(12) - v(1) = 90 - 0 = 90$

P3 $\quad v(123) - v(12) = 220 - 90 = 130$

Marg. Cont $(0, 90, 130)$

There are $3! = 6$ different ways $\qquad 3 \times 2 \times 1 = 3!$

Order of entry			1	2	3
1	2	3	0	90	130
1	3	2	0	120	100
2	1	3	90	0	130
2	3	1	100	0	120

2	3	1	100	0	120
3	1	2	100	120	0
3	2	1	100	120	0
		Total	390	450	480
		Avg	65	75	80
			x_1	x_2	x_3

$$\underbrace{x_1+x_2}_{140} \geq \underbrace{v(12)}_{90}$$

Splitting Savings

	1	2	3	
$c(\cdot)$	100	140	130	
x	65	75	80	
Pay	35	65	50	= 150

Difficulty! $n=15$, $15! > 13\times 10^{11}$

Easier way: $n=10$

Consider #7, and $\{3,5,9\}$

$$v(\{3,5,9,7\}) - v(\{3,5,9\}) = ?$$

$$v(\{3,5,9,\overset{\downarrow}{7}\}) - v(\{3,5,9\}) = ?$$

How many? $\underbrace{\{3,5,9\}}_{3!}$ $\underbrace{7}_{1}$ $\underbrace{6 \text{ other}}_{6!}$

Total # marginal vectors in which 7 gets contribution. So, his total contr. is

$$3!\,6!\,[v(3,5,9,7) - v(3,5,9)]$$

So, Shapley value for 7 is

th

$|S|$: cardinality of S

General:

$$\Phi_i(N,v) = \sum_{\substack{S \subseteq N: \\ i \notin S}} \frac{|S|!\,(n-|S|-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$$

Ex. Three cities $N = \{1,2,3\}$

Ex. Three cities $N = \{1,2,3\}$

Find $\Phi_1(N,v)$ for P1

$$\Phi_1(N,v) = \sum_{\substack{S \subseteq N: \\ 1 \notin S}} \frac{|S|!\,(3-|S|-1)!}{3!}\left[v(S \cup \{1\}) - v(S)\right]$$

$S=\{2\}$: $\frac{1!\,(3-1-1)!}{3!}\left[v(21) - v(2)\right] = \frac{90}{6} = 15$

$S=\{3\}$: $\frac{1!\,(3-1-1)!}{3!}\left[v(31) - v(3)\right] = \frac{100}{6}$

$S=\{2,3\}$: $\frac{2!\,(3-2-1)!}{3!}\left[v(231) - v(23)\right] = \frac{200}{6}$

$\Phi_1(N,v) = 65$

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c) Nucleolus

Schmeidler (1969)

Excess (unhappiness) of a coalition S at x

$$e_S(x) = v(S) - \sum_{i \in S} x_i$$

Ex. $v(1) = v(2) = v(3) = 0$

$v(12) = 60$, $v(13) = 80$, $v(23) = 100$

$v(123) = 105$

Suppose $x = (20, 35, 50)$.

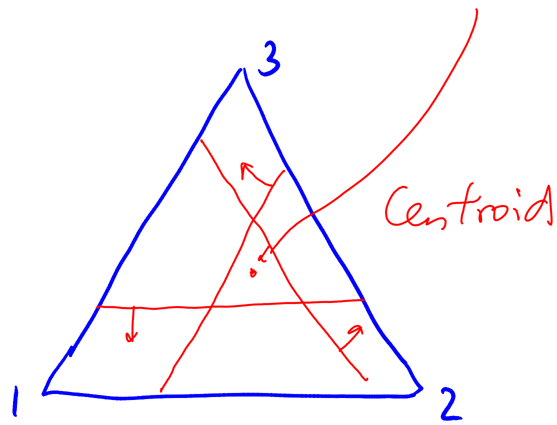
$$e_1(x) = 0 - 20 = -20$$
$$e_2(x) = 0 - 35 = -35$$
$$e_3(x) = 0 - 50 = -50$$
$$\left.\begin{aligned} e_{12}(x) &= 60 - (20+35) = 5 \\ e_{13}(x) &= 80 - (20+50) = 10 \\ e_{23}(x) &= 100 - (35+50) = 15 \end{aligned}\right\} \leftarrow$$
$$e_{123}(x) = 105 - (20+35+50) = 0$$

	12	13	23
e	5	10	15

+5

e	10	10	10	
x	15	35	55	← Nucleolus

Core



Shapley: (25, 35, 45)

S	v(S)	$\sum x_i$	e
1	0	25	-25
2	0	35	-35
3	0	45	-45
12	60	60	0
13	80	70	10
23	100	80	20
123	105	105	0