2012-06-22
Friday, June 22, 2012
9:53 AM

I. Static Games w/ Complete Information (Nash)

Players $1, \dots, n$
Strategies $s_1, \dots, s_n$, $s_i \in S_i$
Payoffs $\pi_1(s_1, \dots, s_n), \dots, \pi_n(s_1, \dots, s_n)$
Normal form $G = \{S_1, \dots, S_n; \pi_1, \dots, \pi_n\}$

1. Examples

Ex. Prisoner's dilemma

		#2 NC	#2 C
#1	NC	1, 1	9, 0
	C	0, 9	6, 6

Def A strategy s_i^* is a best response to a strategy vector s_{-i}^* of other players if

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*) \text{ for all } s_i$$

Def The strategy vector $s^* = (s_1^*, \dots, s_n^*)$ is a

Nash equilibrium (NE) if

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*) \text{ for all } s_i \text{ and } i$$

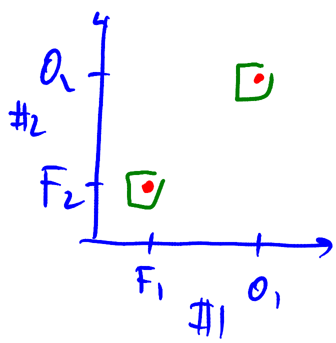
Ex. B of S

$$\begin{array}{ccc} & & W \\ & F_2 & O_2 \\ M \quad \begin{array}{c} F_1 \\ \\ O_1 \end{array} & \multicolumn{2}{c}{\left(\begin{array}{ccc} 2,1 & \leftarrow & 0,0 \\ \uparrow & & \downarrow \\ 0,0 & \rightarrow & 1,2 \end{array}\right)} \end{array}$$

Best response $b_i(\cdot)$, $i=1,2$

$b_1(F_2) = F_1 \qquad b_1(O_2) = O_1$

$b_2(F_1) = F_2 \qquad b_2(O_1) = O_2$



Ex. Pricing problem (Bertrand)

lower price → entire market

Equal " → share "

		F2	
	H_2	M_2	L_2
H_1	6,6	$0,\overline{10}$	0,8

$\cancel{b_1(M_2) \neq L_1}$

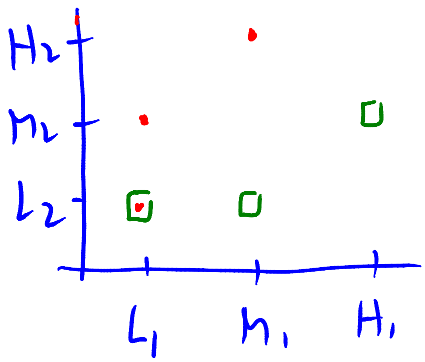
$\cancel{b_2(L_1) = L_2} \neq M_2$

F1	M_1	$\overline{10},0$	$5,5$	$0,\overline{8}$
	L_1	$8,0$	$\overline{8},0$	$\overline{4},\overline{4}$

$b_1(L_2)=L_1$

$b_2(L_1)=L_2$

NE



- Mangasarian + Stone, JMAA '64

$x_{m\times 1}$ $\qquad$ $y_{n\times 1}$

$A_{m\times n}$ $\qquad$ $B_{m\times n}$

$u_{m\times 1}=(1,\dots,1)'$ $\qquad$ $v_{n\times 1}=(1,\dots,1)'$

$$\max_{x,y,a,b} \; x'(A+B)y-a-b$$

$$\text{s.t.} \quad Ay-au\le 0$$

$$B'x-bv\le 0$$

$\sum x_i=1$ —— $u'x=1$

$\sum y_j=1$ $\quad$ $v'y=1$

$$x\ge 0$$

QP

$x \geq 0$

$y \geq 0$

Ex. Continuous strategies

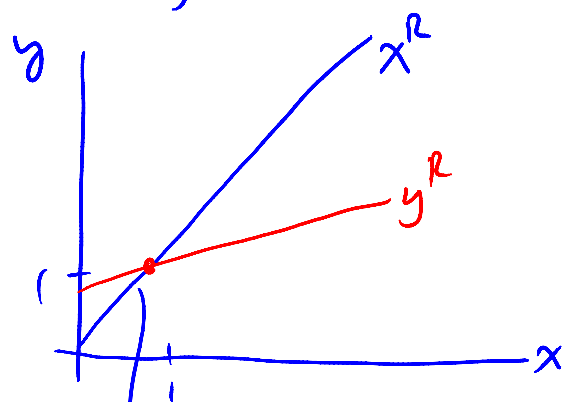
$$f(x,y) = -2x^2 + 5xy$$

$$g(x,y) = -3y^2 + 2xy + y$$

$$x, y \geq 0$$

Given y: $\dfrac{\partial f}{\partial x} = -4x + 5y = 0$: $x^R = b_1(y) = \dfrac{5}{4} y$

" x: $\dfrac{\partial g}{\partial y} = -6y + 2x + 1 = 0$: $y^R = b_2(x) = \dfrac{1}{6}(2x+1)$



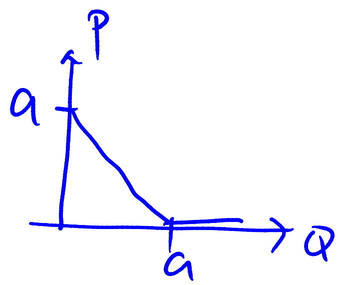
$$(x^*, y^*) = \left(\frac{5}{14}, \frac{4}{14}\right)$$

		P2		
		3/14	4/14	5/14
P1	4/14	;	;	
	5/14			
	6/14			

Ex. Cournot duopoly

	firm 1	firm 2
Quant	q_1	q_2
price	$P(Q) = a - Q$	

$$Q = q_1 + q_2$$



$$c_i(q_i) = c q_i \quad , \; c < a$$

$$S_i = [0, \infty)$$

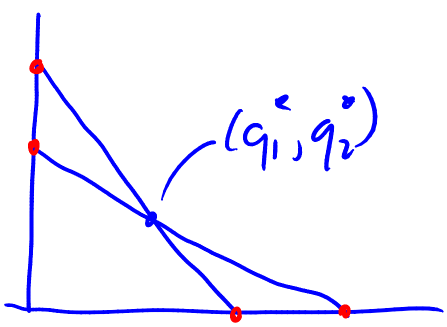
$$\pi_i(q_i, q_j) = q_i \left[P(q_i + q_j) - c \right]$$

$$= q_i \left[a - (q_i + q_j) - c \right]$$

$$\frac{\partial \pi_i}{\partial q_i} = a - \cancel{\tfrac{1}{2}}^{\,2} q_i - q_j - c = 0 \qquad b_i(q_j)$$

$$\Rightarrow \quad \left.\begin{aligned} q_1^* &= \frac{1}{2}(a - q_2^* - c) \\ q_2^* &= \frac{1}{2}(a - q_1^* - c) \end{aligned}\right\} \Rightarrow \underbrace{\boxed{q_1^* = q_2^* = \frac{1}{3}(a-c)}}_{NE}$$

	q_1^*	q_2^*	$\pi_i^*(q_i^*, q_j^*)$
Duopoly	$\frac{1}{3}(a-c)$	$\frac{1}{3}(a-c)$	$\frac{1}{9}(a-c)^2$
Monopoly	$q_M = \frac{1}{2}(a-c)$		$\frac{1}{4}(a-c)^2$



2. Mixed strategies

Ex. Matching pennies

	H	T
H	$-1, \bar{1}$	$\bar{1}, -1$
T	$\bar{1}, -1$	$-1, \bar{1}$

No pure str.

Thm. (Nash '50) For $G=\{S_1,\dots,S_n,\pi_1,\dots,\pi_n\}$ for finite n and finite S_i, $\exists$ Nash equilibrium, possibly involving mixed strategies

Ex. Penny

		(H) q	(T) $1-q$
(H)	p	$-1, 1$	$1, -1$
(T)	$1-p$	$1, -1$	$-1, 1$

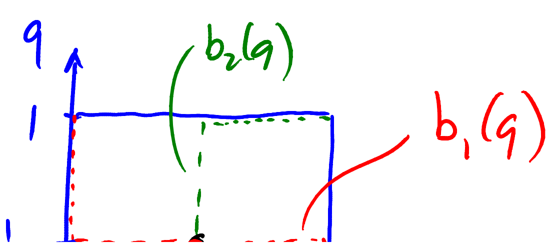
$$v_1(p,q) = -1(pq) + 1(p)(1-q) + 1(1-p)q + (-1)(1-p)(1-q)$$

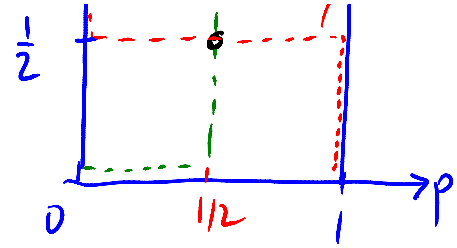
$$= 2p(1-2q) + 2q - 1$$

$1-2q>0$: $\quad 1>2q$, $\quad q<\frac{1}{2} \Rightarrow p=1$

$1-2q<0$: $\quad q>\frac{1}{2} \Rightarrow p=0$

$1-2q=0$ $\quad q=\frac{1}{2} \Rightarrow p\in[0,1]$





$(p^*, q^*) = (\frac{1}{2}, \frac{1}{2})$

$v_1(\quad) = 0$

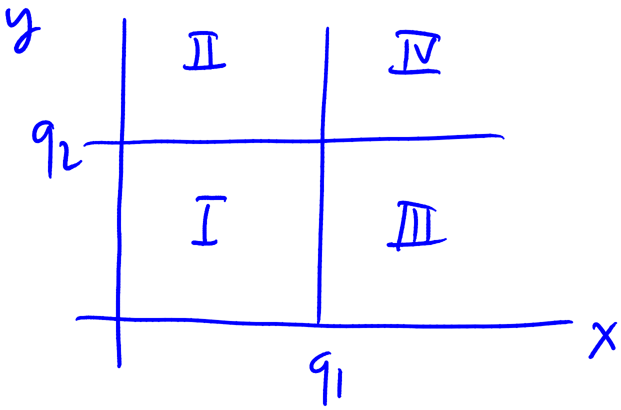
$v_2(\quad) = 0$

3. Two newsvendors

	P1	P2
	X	Y
	$f(x)$	$h(y)$
Sales rev	s_1	s_2
purch cost	c_1	c_2
order quant	q_1	q_2
	$a: P1 \to P2$	$b: P2 \to P1$

$$J_1(q_1, q_2) = s_1 \int_0^{q_1} x f(x)\,dx + s_1 q_1 \int_{q_1}^{\infty} f(x)\,dx$$

$$+ s_1 \int_0^{q_1} \int_{q_2}^{B} b(y - q_2)\, h f \,dy\,dx - c_1 q_1$$



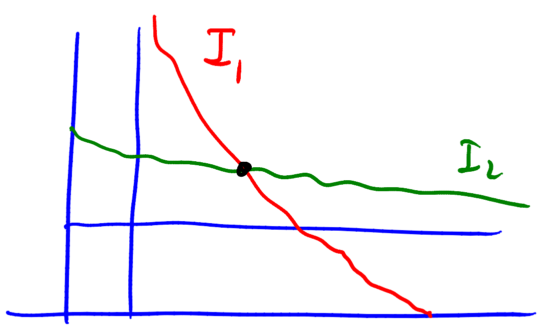
I: π_1^1 = Sales revenue + Salvage − Purch. cost

II: π_1^2 = Direct + indirect sales revenue + Salvage − purch. cost

III: π_1^3 = Sales rev − lost sales penalty − purch. cost

IV: π_1^4 = " − " − "

$$B = (q_1 - x)/b + q_2, \qquad A = (q_2 - y)/a + q_1$$



Parlar '88, Wu + Parlar 2011

$$f(x) = \lambda e^{-\lambda x}, \quad h(y) = \mu e^{-\mu y}, \quad (\lambda, \mu) = \left(\tfrac{1}{30}, \tfrac{1}{20}\right)$$

$(a, b \mid s_1, s_2 \mid c_1, c_2) = (.9, .9 \mid 15, 9 \mid 8, 5)$

$(q_1^*, q_2^*) = (25.38, 19.55)$

$(J_1^*, J_2^*) = (83.63, 35.91)$

4. Existence + Uniqueness issues (Cachon + Netessine '05)

Exist

Thm (Debreu '52): If players' strategy set is compact + convex + payoff functions are cont. + quasi-concave w.r.t. each player's strategy $\Rightarrow \exists$ pure strategy NE.

Uniqueness
- algebraic method
- * - contraction mapping
- univalent mapping
- index Thm

Thm (Contraction mapping).

P1 P2

$x_1 \quad x_2$

$$A = \begin{bmatrix} 0 & \frac{\partial b_1}{\partial x_2} \\ \frac{\partial b_2}{\partial x_1} & 0 \end{bmatrix}$$

where $x_i = b_i(x_{-i})$ is the BR function.

The problem has a <u>unique</u> soln $\Leftrightarrow$

$\rho(A) < 1$ where

$$\rho(A) = \{\max |\lambda| : Ax = \lambda x,\ x \neq 0\}$$

spectral radius

Ex. $f(x,y) = -2x^2 + 5xy$ $\qquad \partial b_1(y)/\partial y = 5/4$

$g \quad = -3y^2 + 2xy + y$ $\qquad \partial b_2(x)/\partial x = 1/3$

$$A = \begin{bmatrix} 0 & 5/4 \\ 1/3 & 0 \end{bmatrix}, \Rightarrow \lambda = \begin{bmatrix} .64 \\ -.64 \end{bmatrix}$$

$\rho(A) = .64 < 1$ $\qquad$ Unique NE